Team Quiz, Saturday, March 4th, 2006

ROUND I

1. A committee of 4 students is to be formed from 5 girls and 7 boys. If the chairperson of this committee must be female and the secretary must be male and the other two members chosen from the remaining students, in how many ways can such a committee be formed?

Answer. There are 5 possible chairwomen and 7 possible secretaries. The other two will be chosen from 10 people. Therefore the answer is

$$\boxed{(5)(7)(45) = 1575.}$$

2. Solve for θ, given that $0 < \theta < 180°$,

$$\cos 5\theta + \cos 3\theta + \cos\theta = 0.$$

Answer. Substitute $\cos 5\theta + \cos\theta = 2\cos 3\theta \cos 2\theta$, so

$$\cos 3\theta(1 + 2\cos 2\theta) = 0.$$

Either

$$\cos 3\theta = 0,$$

so $\theta = 30°$ or $90°$ or $150°$, or

$$\cos 2\theta = -\frac{1}{2},$$

so $2\theta = 120°$ or $2\theta = 240°$. This gives the solutions

$$\boxed{30°, 60°, 90°, 120°, 150°.}$$

ROUND II

3. Solve for x, given that the logarithms are real numbers,

$$\log_5(x + 3) + \log_5(x - 1) = 1.$$

Answer: exponentiate.

$$(x + 3)(x - 1) = 5, \quad \text{so} \quad x^2 + 2x - 8 = 0, \quad \text{or} \quad (x + 4)(x - 2) = 0.$$

Since $x + 3$ and $x - 1$ are both positive,

$$\boxed{x = 2.}$$

4. Evaluate

$$\int_0^1 \frac{e^x}{(1+e^x)^2} dx.$$

Give your answer in the form

$$\frac{a}{b} - \frac{a}{e+a}, \quad a, b \in \mathbb{N}.$$

Answer. Substitute $t = e^x$, then $u = (t+1)^2$:

$$\int_1^e \frac{dt}{(t+1)^2} = \int_1^e \frac{d(t+1)}{(t+1)^2} = \left[-\frac{1}{t+1}\right]_1^e$$

$$\boxed{= \frac{1}{2} - \frac{1}{e+1}.}$$

ROUND III

5. Find the range of values of k such that the equation below has real roots:

$$\frac{5-x}{k} = \frac{k}{x+7}.$$

Give your answer in the form $a \leq k \leq b, \quad a, b \in \mathbb{Z}$.

Answer.

$$(5-x)(x+7) = k^2, \quad \text{so} \quad x^2 + 2x + k^2 - 35 = 0.$$

The solutions of the above equation are real if and only if $4 - 4k^2 + 4(35)$ is nonnegative, so $36 - k^2 \geq 0$ or

$$\boxed{-6 \leq k \leq 6.}$$

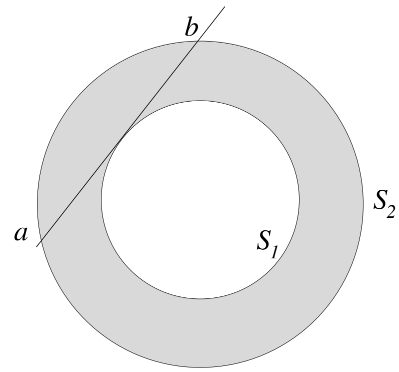
6. The points $A(-8, 6)$ and $B(-6, -8)$ are on the circle with equation $x^2 + y^2 = 100$.

The perpendicular bisector of AB cuts the circle at a point P in the first quadrant and a point Q in the third quadrant. Calculate $|PQ|$.

Answer: the perpendicular bisector of AB passes through the centre, so PQ is a diameter and its length is

$$\boxed{20.}$$

ROUND IV



7. S_1 and S_2 are two concentric circles with centre at c. $[ab]$ is a chord of S_2 and ab is a tangent to the circle S_1. Given that $|ab| = 10$, find the area (shaded) between the two circles.

Your answer is to be in the form $n\pi, \quad n \in \mathbb{N}$.

Answer: let r be the radius of the smaller circle and R the radius of the larger. From Pythagoras

$$r^2 + \left(\frac{|ab|}{2}\right)^2 = R^2,$$

so

$$R^2 - r^2 = 25$$

and the area, $\pi(R^2 - r^2)$, is

$$\boxed{25\pi.}$$

8. Find all the values of $x \in \mathbb{Z}$ such that

$$(x^2 - 3)(x^2 + 5) < 0.$$

Answer: Equivalently, $x^2 - 3 < 0$, so $x^2 < 3$ and

$$\boxed{x = -1, 0, \text{ or } 1.}$$

ROUND V

9. Given that

$$f(x) + 3g(x) = x^2 + x + 6, \quad \text{and} \quad 2f(x) + 4g(x) = 2x^2 + 4,$$

find the values of x for which $f(x) = g(x)$.

Answer. Subtract half the second from the first, getting

$$g(x) = x + 4,$$

and $f(x) = x^2 + x + 6 - 3g(x) = x^2 - 2x - 6$, so $f(x) - g(x) = x^2 - 3x - 10 = (x + 2)(x - 5)$ which vanishes for

$$\boxed{x = -2 \text{ or } 5.}$$

10. Suppose that $\omega^3 = 1$, $\omega \neq 1$, and k is a positive integer. There are two possible values of $1 + \omega^k + \omega^{2k}$, and they belong to $\mathbb{N}$. Find them.

Answer: $k = 1$ gives $1+\omega+\omega^2 = 0$, $k = 2$ gives the same, and $k = 3$ gives $1+\omega^3+\omega^3 = 3$. The answers are

$$\boxed{0 \text{ or } 3.}$$

ROUND VI

11. When $x^3 + px^2 + qx + 1$ is divided by $x - 2$ the remainder is 9; when divided by $x + 3$ the remainder is 19. Find the value of p and the value of q.

Answer: write $f(x)$ for this polynomial. $f(2) = 9$ and $f(-3) = 19$.

$$8 + 4p + 2q + 1 = 9 \quad \text{and} \quad -27 + 9p - 3q + 1 = 19.$$

$$4p + 2q = 0 \quad \text{and} \quad 9p - 3q = 45,$$

so $q = -2p$ and $3p - q = 15$ so

$$\boxed{p = 3 \text{ and } q = -6.}$$

12. Find the value of λ for which

$$3x + 2y + \lambda(x + y + 2) = 0$$

represents the equation of a line perpendicular to the line

$$x - 3y + 1 = 0.$$

Answer: The slope of the second line is 1/3, so the slope of the perpendicular is -3. The slope of the line

$$(3 + \lambda)x + (2 + \lambda)y + 2\lambda = 0$$

is

$$-\frac{3 + \lambda}{2 + \lambda}.$$

Equating this to -3, we get

$$3 + \lambda = 3(2 + \lambda), \quad 2\lambda = -3,$$

so

$$\boxed{\lambda = -3/2.}$$

ROUND VII

13. Evaluate

$$\int_1^2 \frac{dx}{\sqrt{2x - x^2}}.$$

Give your answer in radians.

Answer: substitute $u + 1 = x$ getting

$$\int_0^1 \frac{du}{\sqrt{1-u^2}} = \left[\sin^{-1}(u)\right]_0^1 =$$

$$\boxed{\pi/2.}$$

14. The velocity v cm/sec of a body moving along a straight line is proportional to the square of its distance s from a fixed point O on the line. If $v = 2$ when $s = 10$, find the acceleration when $s = 20$.

Answer: $v = As^2$, and from the data, $A = 1/50$. For the acceleration dv/dt,

$$\frac{dv}{dt} = 2As\frac{ds}{dt} = 2Asv = \frac{sv}{25}.$$

With $s = 20$, $v = 8$, and the acceleration is

$$\boxed{160/25 = 6.4}$$

15. Write

$$\frac{\sin 5A + \sin 3A + \sin A}{\cos 5A + \cos 3A + \cos A}$$

in the form $\tan nA$, $n \in \mathbb{N}$.

Answer:

$$\frac{2\sin 3A \cos 2A + \sin 3A}{2\cos 3A \cos 2A + \cos 3A} = \frac{(2\cos 2A + 1)\sin 3A}{(2\cos 2A + 1)\cos 3A} =$$

$$\boxed{\tan 3A.}$$

16. Find the values of m for which the line $y + mx + 12 = 0$ is a tangent to the circle

$$x^2 + y^2 - 2x - 2y - 6 = 0.$$

Answer: find where the line intersects the circle by substituting $-12-mx$ for y in the equation for the circle, getting

$$x^2 + (12+mx)^2 - 2x - 2(-12-mx) - 6 = (1+m^2)x^2 + 24mx + 144 - 2x + 24 + 2mx - 6 = 0,$$

i.e.,

$$(1+m^2)x^2 + (26m-2)x + 162 = 0.$$

This has repeated roots if

$$(26m-2)^2 - 4(1+m^2)(162) = 0, \quad \text{i.e.} \quad (13m-1)^2 - (1+m^2)(162) = 0,$$

$$169m^2 - 26m + 1 - 162 - 162m^2 = 0, \quad \text{i.e.} \quad 7m^2 - 26m - 161 = 0;$$

$$m = \frac{26 \pm \sqrt{676 + 4508}}{14} = \frac{26 \pm \sqrt{5184}}{14} = \frac{26 \pm 72}{14}.$$

Answers are

$$\boxed{m = 7 \text{ or } -23/7.}$$

ROUND VIII

17. Find the minimum value of

$$\frac{e^x}{x}, \qquad x > 0.$$

Answer: the derivative is

$$\frac{xe^x - e^x}{x^2}$$

which vanishes at $x = 1$ only, so

$$\boxed{\text{the minimum is } e.}$$

18. There exists a function f such that

$$f(x_1 + x_2 + x_3 + x_4 + x_5) = f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5) - 8$$

for all real numbers x_1, x_2, x_3, x_4, x_5. Calculate $f(0)$.

Answer. Let $x = f(0)$. Then

$$f(5 \times 0) = 5f(0) - 8, \quad \text{so} \quad 4f(0) = 8,$$

and

$$\boxed{f(0) = 2.}$$

19. A box contains red marbles and blue marbles. There are 12 more red marbles than blue marbles and the probability of picking a blue marble is 1/4. How many marbles are there in the box?

Answer: Let x be the number of blue marbles, so the total in the box is $2x + 12$. For the probability to be $1/4$, this must equal $4x$, so $x = 6$.

$$\boxed{\text{18 red and 6 blue.}}$$

20. Real numbers a, b, c satisfy the equations

$$a + b + c = 26 \quad \text{and} \quad \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 28.$$

Find the value of

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} + \frac{a}{c} + \frac{c}{b} + \frac{b}{a}.$$

Answer. Multiply $(a+b+c)(1/a+1/b+1/c)$:

$$\frac{a}{a}+\frac{a}{b}+\frac{a}{c}+\frac{b}{a}+\frac{b}{b}+\frac{b}{c}+\frac{c}{a}+\frac{c}{b}+\frac{c}{c}=(26)(28)=728.$$

Replacing $a/a+b/b+c/c$ by 3, and subtracting from both sides, we get the answer:

$$\boxed{725.}$$

TIE-BREAKER

21. Find the value of dy/dx when $xy^2+y-xy=15$, and $x=2, y=3$.
Answer: with implicit differentiation,

$$y^2+2xy\frac{dy}{dx}+\frac{dy}{dx}-y-x\frac{dy}{dx}=0,$$

$x=2, y=3,$

$$9+12\frac{dy}{dx}+\frac{dy}{dx}-3-2\frac{dy}{dx}=0,$$

$$6+11\frac{dy}{dx}=0,$$

so the answer is

$$\boxed{-6/11.}$$

22. Three fair six-sided dice are rolled. What is the probability that not more than one 5 is thrown?

Answer: let a,b,c be the number rolled by each die. Count the throws in which a,b but not c are 5: 5. Similarly for a,c and b,c. In one other result, $a=b=c=5$, more than one 5 is thrown. There are 16 outcomes with more than one 5, so there are 200 with at most one: answer

$$\boxed{25/27.}$$

23. Find the value of k if $k(x^2+2y^2)+(y-2x+1)(y+2x+3)=0$ represents a circle.
Answer. The coefficients of x^2 and y^2 are $2k+1$ and $k-4$, respectively. Equating them,

$$\boxed{k=5.}$$

24. The quadratic

$$x^2-4x-1=2k(x-5)$$

has two equal roots. Calculate the possible values of k.

Answer.

$$x^2-4x-1=2kx-10k=x^2-(4+2k)x+10k-1.$$

This has repeated roots iff

$$(4+2k)^2=4(10k-1), \quad \text{so} \quad (k+2)^2-10k+1=0, \quad \text{or} \quad k^2-6k+5=0.$$

Solutions are

$$\boxed{k=1 \text{ or } 5.}$$