CONTENTS - click to link

Acknowledgement

The IMTA would like to thank The Celtic Press for sponsoring the cost of producing this Newsletter.

IMTA COUNCIL 2006

Chairperson: *Liam Sayers*
Vice-Chair: *Dominic Guinan*
Correspondence: *Mary Gallagher*
Recording Secretary: *Donal Coughlan*
Treasurer: *John McArdle*
Applied Maths: *John Maher*
Newsletter Editor: *Neil Hallinan*
Education Officer, NCCA: *Peter Tiernan*
Third Level: *Donal Hurley, UCC*
Elizabeth Oldham, TCD
Michael Brennan, WIT

Cumann Oidí na h-Éireann
Irish Mathematics Teachers' Association

Founded to promote and assist the teaching of mathematics at all levels.

Membership is open to all those interested in mathematics and mathematics education.

Individual membership may be obtained through a Branch organisation.

Editorial

Welcome to another edition of the IMTA Newsletter as we round off the year of 2006.

In this Newsletter we continue with an international flavour to the contributions. We can read the gentle guidance of Professor Robert P Crease from New York as he identifies the pristine proof of Pythagoras Theorem with the fundamental idea of 'proof' itself—the very foundation-stone of modern mathematics.

On home ground, the contributions from Cork are rather significant. Timely advice on subject inspection from Brendan O'Sullivan is followed up by a thoroughly researched extensive piece on prime numbers. Practically every paragraph forms a jumping-off ground for further investigation or presentation.

Further on, many of you will quickly appreciate the benefits of paying close attention to the method of customising your Microsoft Word Toolbar presented by Horst Punzet. Believe me, it does pay dividends!

And who would have believed that the GAA would be subject to the severe test of mathematics discovered long before the GAA came into being? So if you wish to fill that sticker-album don't really try it alone. Dr. Kevin Hayes and Ailish Hannigan have identified the true cost to the dedicated fan. A very practical application of mathematics.

Thanks to Maurice O'Driscoll who, once again, has given us the benefit of Solutions to the Higher Maths Leaving Cert 2006 in a readily accessible form.

Don't forget the entry form for the Junior Maths Competition and do try to create some magic squares—they appear almost by magic using the quick formula courtesy of Paul Behan.

Sincere thanks to all contributors to this Newsletter.

On other fronts there is the news that the IMTA web-site is active once more and may be accessed at www.imta.ie.

The NCCA has produced a report on its consultative process regarding the future of mathematics teaching and learning. It is available from www.ncca.ie.

The Team Math Final was held in Trinity College in March 2006. The worthy winners were Gonzaga College, Dublin, while the runners-up were Christian Brothers College, Cork. Well done to all students and teachers concerned.

During the summer months significant comments came from the Minister for Education and Science Mary Hanafin, who advocated that Foundation Level Mathematics at Leaving Certificate should be recognised for points. While a number of Third Level colleges indicated that they had points available, it is also worth promoting the idea for many more institutions.

Recently, there has been a very successful first National Maths Week (16th—20th October). This idea started in Waterford Institute of Technology (in conjunction with CALMAST—www.calmast.ie) and quickly caught on around the country—especially in the Third Level colleges. In Dublin, the week started with the Hamilton Walk (16th October) which was very well attended. Groups of students from local schools participated in the Walk and the Cabra Community Group provided welcome refreshments. Many thanks.

It should be noted that 2006 is also the centenary celebration of the birth of Samuel Beckett. A quick search on the internet shows that the language of mathematics is well-embedded in Beckett's exploration of language and communication. So, what are we waiting for?

In the next Newsletter : Length, Digits, Euklid, ...
All contributions are welcome.
Send by e-mail to : hallinann@eircom.net
or St. Mary's, Holy Faith, Glasnevin, Dublin 11.
Neil Hallinan

Branches : Contacts

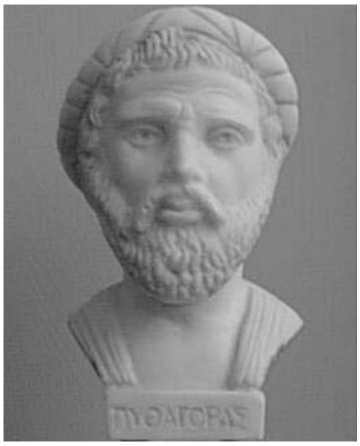
Cork (Sec.): *Brendan O'Sullivan*, bos4@esatclear.ie
Donegal (Sec.): *Joe English*, mathsjc@eircom.net
Dublin (Sec.): *Barbara Grace*, barbara_grace_2004@yahoo.ie
Galway (Tr.): *Mary McMullin*, cmcm@iol.ie
Kerry (Tr.): *John O'Flaherty*, flahjohn@eircom.net
Limerick (Chair): *Gary Ryan*, theboyryan@hotmail.com
Mayo (Sec.): *Lauranne Kelly*, lakelly@stgeraldscollege.com
Midlands (Sec.): *Dominic Guinan*, domguinan@eircom.net
Tipperary (Chair): *Donal Coughlan*, donal.coughlan@esatclear.ie
Waterford (Sec.): *Michael Brennan*, mbrennan@wit.ie
Wexford (Sec.): *Sean McCormack*

The views expressed in this Newsletter are those of the individual authors and do not necessarily reflect the position of the IMTA. While every care has been taken to ensure that the information in this publication is up-to-date and correct no responsibility will be taken by the IMTA for any errors that might occur.

PYTHAGORAS

Pythagoras's theorem changed the life of the British philosopher Thomas Hobbes (1588-1679). Until he was 40, Hobbes was a talented scholar exhibiting modest originality. Versed in the humanities, he was dissatisfied with his erudition, and had little exposure to the exciting new breakthroughs achieved by Galileo, Kepler and other scientists who were then revolutionizing the scholarly world.

One day, in a library, Hobbes saw a display copy of Euclid's *Elements* opened to Book I Proposition 47, Pythagoras's theorem. He was so astounded by what he read that he used a profanity that his first biographer, John Aubrey, refused to spell out: " 'By G__' Hobbes swore, 'this is impossible!'." He read on, intrigued. The demonstration referred him to other propositions, and he was soon convinced that the startling theorem was true.



ΠΥΘΑΓΟΡΑΣ

Hobbes was transformed. He began obsessively drawing figures and writing calculations on bed sheets and even on his thigh. His approach to scholarship changed. He began to chastise philosophers of the day for their lack of rigour and for being unduly impressed by their forebearers. Hobbes compared other philosophers unfavourably with mathematicians, who proceeded slowly but surely from "low and humble principles" that everyone understood.

In books such as *Leviathan*, Hobbes reconstructed political philosophy by establishing clear definitions of terms, then working out implications in an orderly fashion. Pythagoras's theorem had taught him a new way to reason and to present persuasively its fruits.

Before Pythagoras

Pythagoras's theorem is important for its content as well as for its proof. But the fact that lines of specific lengths (3, 4 and 5 units, say) create a right-angled triangle was empirically discovered in different lands long before Pythagoras. Another empirical discovery was the rule for calculating the length of the long side of a right triangle (c) knowing the lengths of the others (a and b), namely $c^2 = a^2 + b^2$.

Indeed, a Babylonian tablet from about 1800 BC shows that this rule was known in ancient Iraq more than 1000 years before Pythagoras, who lived in the sixth century BC. Ancient Indian texts accompanying the Sutras, from between 100 and 500 BC but clearly passing on information of much earlier times, also show a knowledge of this rule. An early Chinese work suggests that scholars there used the calculation at about the same time as Pythagoras, if not before.

But what we do not find in these works are proofs - demonstrations of the general validity of a result based on first principles and without regard for practical application. Proof was itself a concept that had to be discovered. In Euclid's *Elements* we find the first attempt to present a more or less complete body of knowledge explicitly via proofs.

Euclid does not mention Pythagoras, who lived some 200 years previously, in connection with Proposition 47. We credit it to Pythagoras on the authority of several Greek and Latin authors, including Plutarch and Cicero, who wrote half a millennium after Pythagoras. These authors seem to be relying, in turn, on a single source - a certain Apollodorus - about whom next to nothing is known. Apollodorus does not even show how Pythagoras originally proved the theorem.

Pythagoras's theorem is unique for the peculiar way in which it has become a challenge to devise new proofs for it. These proofs are not necessarily any better; most rely on the same axioms but follow different paths to the result. Leonardo da Vinci, Christiaan Huygens and Gottfried Leibniz contributed new proofs. So did US Congressman James Garfield in 1876, before he became the 20th US president.

Indeed, more than a dozen collections of proofs of Pythagoras's theorem have appeared. In 1894 the *American Mathematical Monthly* began publishing proofs, but stopped after about 100. That did not prevent one reader - a teacher from Ohio called Elisha S Loomis - from publishing a book with 230 proofs in 1927; its second edition in 1940 contained 370. The *Guinness Book of World Records* website, under "Most proofs of Pythagoras's theorem", names someone who, it is claimed, has discovered 520 proofs.

The appeal of the theorem

One may wonder what there is to gain by proving a theorem over and over again in different ways. The answer lies in our desire not merely to discover, but to view a discovery from as many angles as possible. But what is it that is so fascinating about Pythagoras's theorem in particular? First, the theorem is important. It helps to describe the space around us and is essential not only in construction but - suitably adapted - in equations of thermodynamics and general relativity. Second, it is simple. The Hindu mathematician Bhaskara was so enamoured of the visual simplicity of one proof that he redid it as a simple diagram - and instead of an explanation wrote a single word of instruction: "See".

Third, it makes the visceral thrill of discovery easily accessible. In an autobiographical essay, Einstein wrote of the "wonder" and "indescribable impression" left by his first encounter with Euclidean plane geometry as a child, when he proved Pythagoras's theorem for himself based on the similarity of triangles. "[F]or anyone who experiences [these feelings] for the first time," Einstein wrote, "it is marvellous enough that man is capable at all to reach such a degree of certainty and purity in pure thinking."

The critical point

Small wonder that Pythagoras's theorem became a model of what a proof is and does. In Plato's dialogue *Meno*, for instance, Socrates coaxes a slave boy (ignorant of geometry) to prove a simplified version of the theorem: that the area of the square formed on the diagonal connecting the corners of another square is twice the area of the first square. Socrates leads the boy to see the inadequacy of the obvious answers, provoking bewilderment and curiosity. Then he helps the boy to recast the problem within a larger, richer context where the path to the solution is clear. Socrates does this exercise not to educate the slave boy, but to illustrate to his owner what learning is all about.

For Hobbes and countless others, Pythagoras's theorem was far more than a means to compute the length of hypotenuses. It shows something more, the idea of proof itself. It provides what philosophers call categorical intuition; it reveals more than a bare content but a structure of reasoning itself. It is a proof that demonstrates Proof.

Robert P Crease
Department of Philosophy
State University of New York at Stony Brook, and historian at the Brookhaven National Laboratory
e-mail rcrease@notes.cc.sunysb.edu

Reprinted from www.physicsweb.org by kind permission of the author and publisher. (Ed.)

WISE WORDS

By I. M. A. Worm

In mathematics, as in music, a great introduction is worth a hundred lessons. (That’s me trying to be Beethoven!)

Only mathematicians understand the world, the rest of us enjoy it!

Only professors use mathematics, the rest of us use a tape-measure and common sense!

If only I were famous, then I would teach mathematics differently. I would start at the end and work my way back through to the beginning and then everyone would understand what they were being taught and suddenly they would realise that most of them never needed to be taught at all …

Preparing for the Subject Inspection

With the implementation of the Education Act, we have gradually seen the extension of the inspectorate and they are certainly making their presence felt in our schools. It is envisioned that every post primary school will have a Whole School Inspection every ten years or so. In addition to this, regular subject inspections are expected to take place. When initial contact is made with the school, it is natural that the prospect is greeted with some fear and anxiety. Many have not had another outside presence in their room with thirty years or more. However, it must be remembered that part of the inspector's role is to support and assist the teachers in carrying out their duty. While we can all become obsessed with the concept of reporting and we can view our situations using a pessimistic lens, it is vital that we concentrate on the good work that is constantly completed and the exacting standards that are aspired to and reached. It is useful to consider some aspects of teaching life that will be subjected to scrutiny.

The first instance is that of the mathematics department. It is important that an actual department exists within a school and some individual has taken on the role of coordinating its activities. A department head should have records of each year group and which teachers have been assigned to them. Agreed topics arising from the initial meeting at the start of the year should all be recorded. While teachers can approach topics in different sequences, it is important that the department head is aware of how the subject is being approached in each year. Of particular note, is how Transition Year is approached in a school. It is here that teachers have the freedom to design their own topics for instruction. Teachers can do much to instill a love of mathematics and provide exposure to mathematical topics that students would not have access to otherwise. While the teacher will have to adjust and amend material according to the class group, it is important that the department head should be provided with a description of the material that will be covered in Transition Year. Finally, the department head should be able to provide the inspector with access to the shared resources that the school has. Ideally, the school should have provided space for the subject department so that teaching resources can be stored safely. Material used in classes should be shared and made available to any new members who join the department as time goes by.

The actual inspection itself takes place over either a single or double class period. After entering the school, the inspector meets with all individuals teaching mathematics in the school and he or she generally outlines how the inspection will take place. Before the visit, the inspector will have received copies of timetables and will plan the most effective way to complete the visit. Every teacher of the subject can expect to be visited and the inspector will cover each year group during this time. It is at this stage that you can raise any questions that might be causing you trouble! Having completed this group meeting, the inspector will then set about completing the schedule that they have determined.

Before the actual lesson, you should take a few moments to set the class in context for the inspector. Discuss the topic that you intend to teach and mention matters such as ability, homework, assessment and any other matters that you might consider relevant. Remember that no one knows your class better than you and you should take the opportunity of allowing the inspector to understand the particular approach that you have adopted with your students.

During the actual inspection, the inspector will sit in the room and complete a form entitled SI – 02. It has the more formal title of 'Record of Evidence for the Quality of Learning and teaching of the Subject'. It is during this time that you will conduct your class as normal and the inspector will make notes on what he or she can observe. Due to extensive training, the inspector will be able to discern a number of things in a relatively short time. Classroom atmosphere will be observed and the manner in which you monitor homework will be noted. During the course of your lesson, your students' work will be looked at, as well as any planning documentation that you might provide. The rest will concentrate on the actual lesson that you conduct and the methodologies that you apply. The inspector will not only record what takes place but will also analyze it and conclude as to whether it is providing adequate teaching and learning of mathematics.

The most important thing is to remember to stay calm and embrace the inevitable! You will more than likely find that it is an invigorating experience and aspects of your work that had gone uncelebrated will now receive the recognition that they deserve. Very often, the inspection can bring members of a department closer together and it can provide an impetus for greater co-operation in the future. Teachers can carefully consider how they approach their lessons and consider new paths for the future.

Brendan O'Sullivan
Cork

USEFUL WEBSITES

www.mathssupport.ie This site has been set up by the Mathematics Support Service. It has a Discussion Forum to answer teachers' questions online. It also has a Newsletter!

www.examinations.ie Home of the State Examinations Commission. Marking schemes for examinations; Chief Examiners' Reports; ...

Primes

Introduction

This work aims to demonstrate the importance of prime numbers within both the theory of numbers and mathematics itself. There are many rich areas contained within these pages that students can easily be introduced to, allowing them to further feed their interest in the study of mathematics generally. The teacher can consider prime numbers as a unit within Transition Year or take any of the topics that will follow and use it as a springboard for further exploration by enquiring minds.

To provide firm foundations, I will give definitions of terms that are used throughout the work. Having made this clarification, the reader will be taken on a journey through history establishing how primes were valued and hopefully gain an idea of the mathematical work that was done around this topic.

The contributions of people such as Euler, Gauss and Fermat will all be acknowledged. These colossal figures are worthy subjects that students can further research and produce projects on, thus deepening their appreciation for the history of mathematics.

I will detail the ongoing search for the largest primes in existence, as well as the search for the largest primes of a certain type such as Mersenne primes and Fermat primes.

Of a practical nature, I will outline how primes are of importance commercially, being used in the process of encryption and I will demonstrate how primes are used within the RSA cryptosystem.

Finally, I will discuss the open questions concerning primes that continue to fascinate mathematicians around the world.

Definitions

An integer $p > 1$ is called a prime number, or simply a prime, if its only positive divisors are 1 and p. An integer greater than 1, which is not a prime, is termed composite. For example, 19 is a prime number since there are no numbers other than 1 and 19 that divide it evenly. On the other hand, 39 is a composite number because 3 divides 39 evenly ($39 \div 3 = 13$).

Among the first ten positive integers 2, 3, 5 and 7 are all primes, while 4, 6, 8, 9 and 10 are composite numbers. 2 is the only even prime number, all other even numbers are divisible by 2 and so are not prime. The integer 1 is considered to be neither prime nor composite. [1]

Ancient Times

Prime numbers and their properties were first studied extensively by the ancient Greek mathematicians. The mathematicians of Pythagoras' school (500 BC to 300 BC) were interested in numbers for their mystical and numerological properties. Its arithmetic was a highly speculative science which had little in common with the contemporary Babylonian computational technique. Numbers were divided into classes: odd, even, even-times-even, odd-times-odd, prime and composite, perfect, amicable, triangular, square, pentagonal and more. To look at one specific classification, a perfect number is one whose proper divisors sum to the number itself. The numbers 6 and 28 are such numbers. The number 6 has proper divisors 1, 2 and 3 and 1 + 2 + 3 = 6. 28 has divisors 1, 2, 4, 7 and 14 and 1 + 2 + 4 + 7 + 14 = 28. A nice exercise is to set your students the task of finding other perfect numbers! The Pythagoreans investigated the properties of numbers, adding their brand of number mysticism and placing them in the centre of a cosmic philosophy which tried to reduce all fundamental relations to number relations ("everything is number"). It is the Ancient Greek mathematicians that must be thanked for the classification of numbers as prime and composite. [2]

By the time Euclid's Elements appeared around 300 BC, several important results about primes had been proved. Proposition 14 of Book IX of Euclid's elements contains the result which later became known as the Fundamental Theorem of Arithmetic. This states that every integer greater than 1, except for the order of the factors, can be represented as a product of primes in one and only one way. For example, $6 = 2\times3$. This is called a number's prime factorization. Several of the primes which appear in the factorization of a given positive integer may be repeated as is the case with $360 = 2\times2\times2\times3\times3\times5$. It is this theorem that provides the reason why 1 is not considered to be a prime. If 1 were prime, the Fundamental Theorem of Arithmetic would be false. For example, 12 = $2\times2\times2$ could now be written as $1\times2\times2\times2$ or $1\times1\times2\times2\times3$. So the representation would no longer be unique. [3] This is a useful way of demonstrating to students the special position that 1 has amongst numbers.

Around 200 BC the Greek Eratosthenes devised an algorithm for calculating primes called the Sieve of Eratosthenes. The use of this "sieve" allows the finding of all primes below a given integer n. The scheme calls for writing down the integers from 2 to n in their natural order and then systematically eliminating all the composite numbers by striking out all multiples p, 2p, 3p,… of the primes $p \leq \sqrt{n}$. The integers that are left on the list – those that do not fall through the "sieve" – are primes. [4]

To see an example of how this works, suppose that we wish to find all primes not exceeding 100. Consider the sequence of consecutive integers 2, 3, 4 …,100. Recognising that 2 is a prime, we begin by crossing out all the even integers from our listing, except 2 itself. The first of all the remaining integers is 3, which must be a prime. We keep 3, but strike out all higher multiples of 3, so that 9, 15, 21, … are now removed. The smallest integer after 3 which has not yet been deleted is 5, another prime. We next remove its multiples. After eliminating the proper multiples of 7, the guide $\sqrt{100} = 10$ tells us that all composite integers have fallen through the sieve. It is not necessary to travel beyond the multiples of 7 as we have deleted all the multiples of the primes less than 10. You can extend the range and allow students to see just how accurate the sieve can be. It sometimes proves to be quite competitive!

Worked Example using the Sieve of Eratosthenes

	02	**03**	~~04~~	**05**	~~06~~	**07**	~~08~~	**09**	~~10~~
11	~~12~~	**13**	~~14~~	~~15~~	~~16~~	**17**	~~18~~	**19**	~~20~~
~~21~~	~~22~~	**23**	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	**29**	~~30~~
31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	**37**	~~38~~	~~39~~	~~40~~
41	~~42~~	**43**	~~44~~	~~45~~	~~46~~	**47**	~~48~~	~~49~~	~~50~~
~~51~~	~~52~~	**53**	~~54~~	~~55~~	~~56~~	~~57~~	~~58~~	**59**	~~60~~
61	~~62~~	~~63~~	~~64~~	~~65~~	~~66~~	**67**	~~68~~	~~69~~	~~70~~
71	~~72~~	**73**	~~74~~	~~75~~	~~76~~	~~77~~	~~78~~	**79**	~~80~~
~~81~~	~~82~~	**83**	~~84~~	~~85~~	~~86~~	~~87~~	~~88~~	**89**	~~90~~
~~91~~	~~92~~	~~93~~	~~94~~	~~95~~	~~96~~	**97**	~~98~~	~~99~~	~~100~~

Proof of the Infinity of Primes

Having completed this task, there is an obvious question that presents itself. Is there a largest prime number, or do the primes go on forever? Euclid presents his answer in a remarkably simple proof in Book IX of his elements. To put it plainly: Given any finite list of prime numbers, one can always find a prime not on the list, hence the number of primes is infinite.

To understand this in a mathematical way, suppose we have a complete list of all the prime numbers up to p. Consider the integer N= $(2 \times 3 \times 5 \times ... p) + 1$, which is one greater than the product of all the primes up to p. When N is divided by any prime number between 2 and p, it leaves a remainder of 1. Now if N is a prime number, it is a prime greater than p, since N is greater than p. If it is not a prime, it may be factored into prime numbers. However, none of its prime factors can be between 2 and p. Either way, there is a prime number greater than p. Therefore, the list of primes never ends. [5]

While there is an endless list of primes, their distribution within the positive integers is quite puzzling. Repeatedly in their distribution there are hints or shadows of a pattern. However an actual pattern that can be described precisely remains to be found. The difference between consecutive primes can be small as with the pairs 11 and 13. At the same time there exists long intervals in the sequence of integers which are totally devoid of any primes. In fact, it is arbitrarily long: Given any N, consider the sequence (N+1)! + 2, … , (N+1)!+ N+1 and note that (N+1)!+k is divisible by k, so this is a sequence of N consecutive composites.

Modern Times

In 17th century France, mathematicians started looking at the set of prime numbers in the hope of identifying patterns that could be exploited. At this time, there were no scientific periodicals in existence. This lead to the formation of discussion groups and constant correspondence between those working on scientific problems. Some figures gained merit by serving as a focus of scientific interchange. One of the best known of these individuals is Father Marin Mersenne, whose name as a mathematician is preserved in "Mersenne numbers". He conjectured that numbers of the form $2^p - 1$ were prime when p = 2, 3, 5, 7, 13, 17, 19, 31, 67, 127 and 257.

2^2—1 = 4—1 = 3
2^3—1 = 8—1 = 7
2^5—1 = 32—1 = 31
2^7—1 = 128—1 = 127
2^{13}—1 = 8192—1 = 8191
2^{17}—1 = 131702—1 = 131701
2^{19} –1 = 524288—1 = 524287
2^{31}—1 = 2147483648—1 = 2147483647
2^{67}—1 = 147573952589676412928—1 = 147573952589676412927
2^{127}—1 = 170141183460469231731687303715884105728—1 =
170141183460469231731687303715884105727
2^{257}—1 = 231584178474632390847141970017375815706539969331281128078915168015826259279872—1 =
231584178474632390847141970017375815706539969331281128078915168015826259279871

He was wrong about 67 and 257,
For example, 2^{67}—1 = 147573952589676412927 = 761838257287 *x* 193707721

He also missed 61, 89 and 107.

$$2^{61} - 1 = 2305843009213693952 - 1 = 2305843009213693951$$
$$2^{89} - 1 = 618970019642690137449562112 - 1 = 618970019642690137449562111$$
$$2^{107} - 1 = 162259276829213363391578010288128 - 1 = 162259276829213363391578010288127$$

Students can exercise their calculator skills by testing the validity of Mersenne's conjectures; some computer software would be required for those that extend beyond ten digits!

His work in this area offered a great legacy. For many years numbers of this form provided the largest known primes. The number M_{19} was proved to be prime by Cataldi in 1588 and this was the largest known prime for about 200 years until Euler proved that M_{31} was prime. This established the record for another century until Lucas showed that M_{127} was prime. Further progress was not made until the arrival of the computer. In 1952 the Mersenne numbers $M_{521}, M_{607}, M_{1279}, M_{2203}$ and M_{2281} were proved to be prime by Robinson using an early computer. By 2001, a total of 39 Mersenne primes had been found. The largest one being $M_{13466917}$. While Mersenne was not the greatest mathematician, he began a search that has spanned nearly four centuries and continues to this day. [6]

Descartes, Fermat, Desargues, Pascal and many other mathematicians corresponded with Mersenne. It was said of him that *"to inform Mersenne of a discovery, meant to publish it throughout the whole of Europe."* You might be familiar with similar figures in your staff room! In the area of primes, Fermat produced a lot of useful work. Fermat, who actually made his living as a lawyer, created numerous conjectures about prime numbers. Just as Mersenne worked with certain types of numbers, Fermat did the same. Numbers of the form $F_n = 2^{2^n} + 1$ are called Fermat numbers and those which are prime are called Fermat primes. Fermat conjectured that F_n is prime for every n $\geq$ 0.

$F_0 = 2^{2^0} + 1 = 2^1 + 1 = 2 + 1 = 3$

$F_1 = 2^{2^1} + 1 = 2^2 + 1 = 4 + 1 = 5$

$F_2 = 2^{2^2} + 1 = 2^4 + 1 = 16 + 1 = 17$

$F_3 = 2^{2^3} + 1 = 2^8 + 1 = 256 + 1 = 257$

$F_4 = 2^{2} + 1 = 2^{16} + 1 = 65536 + 1 = 65537$

For n=0,…,4 the numbers F_n =3, 5, 17, 257, 65537 are indeed prime, but in 1732 Euler showed that the next Fermat number

$F_5 = 2^{2^5} + 1 = 2^{32} + 1 = 4294967296 + 1 = 4294967297$ is composite.
It has factors 641 *x* 6700417.

The Fermat numbers have been studied intensively, often with the aid of computers, but no further Fermat primes have been found. It is conceivable that there are further Fermat primes, even an infinite number of them perhaps. However, they have not been found yet and little headway has been made in this area. Complete factorizations are known only for $F_6, F_7, F_8, F_9, F_{10}$ and F_{11}. Brent found the factorization of F_{11} in 1989 and even with the fastest computers at their disposal, progress in this area is quite slow. These primes are important in geometry. In 1801 Gauss showed that a regular polygon with k sides can be constructed by ruler-and-compass methods if and only if $k = 2^e p_1 \ldots p_r$ where $p_1, \ldots, p_r$ are distinct Fermat primes. [7]

Fermat's Little Theorem

During his time, Fermat proved a speculation of Albert Girard that every prime number of the form 4n+1 can be written in a unique way as the sum of two squares. For example, 13 = 4 + 9, 4 being the square of 2 and 9 being the square of 3. Fermat also provided what has come to known as Fermat's Little Theorem: If p is prime and a is a positive integer with p not divisible by a then $a^{p-1} \equiv 1 \bmod p$. For example let p=3 and let a=4. Using these values, $4^2 = 16$ and 16 $\equiv 1 \bmod 3$. This theorem is the basis for many other results in Number Theory and is the basis for methods of checking whether numbers are prime that are still in use on today's computers. [8] It is also a nice way to get your students working on indices and congruency!

Euler was one of the most prolific mathematicians of all time. He made important contributions in every field of mathematics that existed in his day. Number theory and the primes were no exception. He was the first to factorize the 5th Fermat number $2^{32}+1$. Along a more original vein, he produced the polynomial $f(n) = n^2 + n + 41$, whose values are prime numbers for n = 0, 1, …,39. For example, suppose we let n =36, this produces $(36)^2 + 36 + 41 = 1373$, a prime number. This is known as the trinomial of Euler. Just like Fermat and Mersenne, Euler also has a type of prime named after him. They are also known as symmetric primes due to their distribution in relation to each other. Euler primes describe every pair of primes p, q which are symmetrically placed about $\frac{p+q}{2}$. For example, 3 and 11 are both 4 units away from the number 7. Hence 3 and 7 are symmetric primes. Notice that all twin primes must be symmetric primes. All integers above 3 seem to have an Euler prime. It is also interesting to note that they are not exclusive. For example, the number 7 can be treated as an Euler prime with a number of other primes. 7 and 11 are symmetric about the number 9, 7 and 13 are symmetric about the number 10, 7 and 19 are symmetric around the number 13, finally 7 and 31 are symmetric around the number 19. [9]

The Totient Function of Euler

Euler's phi function was another important development. Euler's phi (or totient) function of a positive integer n is the number of integers in {1, 2, 3, …,n} which are relatively prime to n. Two integers are relatively prime if there is no integer greater than one that divides them both, i.e. their greatest common divisor is one. For example, 12 and 13 are relatively prime but 12 and 14 are not. A list of integers is mutually relatively prime if there is no integer that divides them all. For example, the integers 30, 42, 70 and 105 are mutually relatively prime, but not pair-wise relatively prime. Euler's phi function is usually denoted $\phi(n)$.

The table below outlines Euler's phi function for the first 16 positive integers:

Integer n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\phi(n)$	1	1	2	2	4	2	6	4	6	4	10	4	12	6	8	8

Clearly for primes p, $\phi(p)$ = p-1. Since $\phi(x)$ is a multiplicative function, its value can be determined from its value at the prime powers. [10] Students often say that they cannot fully grasp the concept until they have verified the results for themselves on the calculator!

This leads us on to Euler's extension of Fermat's Little Theorem. Euler's theorem states that if gcd (a, n) = 1, then $a^{\phi(n)} = 1(\mathrm{mod}\ n)$. When n is a prime, this theorem is just Fermat's Little Theorem. For example, $\phi(12)$ =4, so if gcd(a, 12) = 1, then $a^4 = 1\ (\mathrm{mod}\ 12)$. The set of residue classes {d mod n | gcd (d, n) = 1} modulo n form a multiplicative group, so Euler's theorem is a special case of Lagrange's theorem: the order of an element divides the order of a group. [11]

Euler was the first to realize that number theory could be studied using the tools of analysis and in so doing founded the subject of Analytic Number Theory. He was able to show that the series $1/2 + 1/3 + 1/5 + 1/7 + 1/11 + 1/13 + \cdots$ was divergent. This series is formed by summing the reciprocals of the prime numbers, Euler was able to demonstrate that it diverges very slowly.

The Goldbach Conjecture

In 1742, Goldbach wrote a letter to Euler and raised a question concerning primes that remains to be answered to this day. He suggested that every integer greater than 5 is the sum of three primes. Euler replied that this is equivalent to every even integer greater than 2 is the sum of two primes. So Goldbach's conjecture is now known as: Every even n > 2 is the sum of two primes. It is easy to confirm for the first few even integers:

2 = 1 + 1
4 = 2 + 2 = 1 + 3
6 = 3 + 3 = 1 + 5
8 = 3 + 5 = 1 + 7
10 = 3 + 7 = 5 + 5
12 = 5 + 7 = 1 + 11

14 = 3 + 11 = 7 + 7 = 1 + 13
16 = 3 + 13 = 5 + 11
18 = 5 + 13 = 7 + 11 = 1 + 17
20 = 3 + 17 = 7 + 13 = 1 + 19

Euler replied to Goldbach: *"That every even number is a sum of two primes, I consider an entirely certain theorem in spite of that I am not able to demonstrate it."* It seems that Euler never seriously tried to prove the result, but, writing to Goldbach at a later date he countered with a conjecture of his own: any even integer greater than or equal to 6 of the form 4n+2 is a sum of two numbers each being either primes of the form 4n+1 or 1. Schnizel showed that Goldbach's conjecture is equivalent to every integer n > 17 is the sum of three distinct primes. Ramare proved that every even integer is the sum of at most six primes and in 1966 Chen proved every sufficiently large even integer is the sum of a prime plus a number with no more than two prime factors. In 1993 Sinisalo verified Goldbach's conjecture for all integers less than 4×10^{11}. Deshouillers, Saouter and te Riele have verified this up to 10^{14} with the help of a Cray C90 and various workstations. In 1998, Richstein completed verification to 4×10^{14}. More recently in 2003, Oliveira e Silva verified it for 6×10^{16}. Despite all such calculation, a general proof or counterexample is still awaited. The majority of mathematicians believe the conjecture to be true, mostly based on statistical considerations focusing on the probabilistic distribution of prime numbers: the bigger the even number, the more "likely" it becomes that it can be written as a sum of two primes. [12]

An Unclaimed Prize

We know that every even number can be written as the sum of at most six primes. As a result of work by Vinogradov, every sufficiently large even number can be written as the sum of at most four primes. Vinogradov proved furthermore that almost all even numbers can be written as the sum of two primes (in the sense that the fraction of even numbers which can be so written tends towards 1). In 1966, Chen Jing-run showed that every sufficiently large even number can be written as the sum of a prime and a number with at most two prime factors. In 1982 Doug Lenat's Automated Mathematician independently rediscovered Goldbach's Conjecture in one of the earliest demonstrations that Artificial Intelligences were capable of scientific discovery. [13] In order to generate publicity for the book *Uncle Petros and Goldbach's Conjecture* by Apostolos Doxiadis, publishers Faber and Faber offered a $1,000,000 prize for a proof of the conjecture in 2000. The prize was only to be paid for proofs submitted for publication before April 2002. The prize was never claimed. [14] The mention of this kind of money does much to focus the mind of budding mathematicians!

Wilson's Theorem

Another interesting development in number theory relating to primes came in the later half of the 18th century. Wilson's theorem was first discovered by John Wilson, a student of the English mathematician Edward Waring. Waring announced the theorem in 1770, although neither could prove it. Lagrange gave the first proof in 1771. There is evidence that Leibniz was aware of the result a century earlier, but he never published it. [15]

Wilson's theorem states that a number n > 1 is prime if and only if (n-1)! $\equiv -1 \pmod{n}$.

For example, let n = 5

By Wilson's theorem:

5 is prime iff $(5-1)! \equiv -1 (\bmod 5)$

4!=24 and 24 $\equiv -1 (\bmod 5)$ and so 5 must be prime. Wilson's theorem is both necessary and sufficient for primality. This beautiful result is of mostly theoretical value though because it is relatively difficult to calculate (p-1)!, making it of little use for testing primality. In contrast it is easy to calculate a^{p-1} , so elementary primality tests are built using Fermat's Little Theorem rather than Wilson's Theorem. [16] For example, the largest prime ever shown prime by Wilson's theorem is most likely 1099511628401. Even with a clever approach to calculating n!, this still takes about one day on a SPARC processor. On the other hand, numbers with tens of thousands of digits have been shown to be prime using a converse of Fermat's little theorem in less than an hour.

An interesting corollary to Wilson's theorem involves trigonometry:

The integer n is prime if and only if $\sin \dfrac{(((n-1)!+1)\pi)}{n}$ =0 [17]

Let n=5

5 is prime iff $\sin \dfrac{(((5-1)!+1)\pi}{5} = 0.$

4!=24 results in $\sin \dfrac{25\,\pi}{5} = \sin\, 5\pi \,=\, 0\,.$

So 5 is a prime number. I think this is a nice way for students to bring different areas of mathematics together simply.

Just as Mersenne, Fermat and Euler have had the honour of having primes named after them, Wilson is also connected with primes that satisfy a certain condition. It must be said that there are very few Wilson primes.

A Wilson prime is a prime that satisfies

p^2 divides (p-1)! + 1

For example 5 is a Wilson prime because
25 divides 4! + 1 = 25.

In 1997, Crandall, Dilcher and Pomerance verified that 5, 13 and 563 are the only Wilson primes less than 5×10^8. It is conjectured that the number of Wilson primes is infinite and that the number of such primes between x and y should be about $\log(\frac{\log y}{\log x})$. With this kind of conjecture, it might be quite some time before the fourth such prime is found. [18] There is also a Wilson composite—this is provided by an analog of Wilson's theorem that applies to composites:

> Let n be an integer that is greater than one. Let m be the product of all the positive integers less than n, but relatively prime to n (so m = (n-1)! if n is prime), n divides either m + 1 or m – 1. We say a composite number n is a Wilson Composite if n^2 divides either m + 1 or m – 1.
>
> The only such number below 50000 is 5971. Others include 558771, 1964215, 8121909 and 12326713; there are no other less than 10,000,000. [19]

The Legacy of Gauss

Gauss, one of the greatest mathematicians in history, once said the following about finding an easy way to find prime numbers: *" The problem of distinguishing prime numbers from composite numbers and of resolving the latter into their prime factors is known to be one of the most important and useful in arithmetic. It has engaged the industry and wisdom of ancient and modern geometers to such an extent that the dignity of the science itself seems to require that every possible means be explored for the solution of a problem so elegant and so celebrated."* Gauss, who could be described as a compulsive calculator, told a friend that whenever he had a spare 15 minutes he would spend it by counting the primes in a 'chiliad' (a range of 1000 numbers). By the end of his life it is estimated that he had counted all the primes up to about 3 million. [20]

His left a more important legacy than this though, Euler stated the Quadratic Reciprocity Theorem in 1783 without proof. Legendre was the first to publish a proof, but there were errors within it. In 1796, at the age of nineteen, Gauss became the first to publish a correct proof. The quadratic reciprocity theorem was Gauss' favourite theorem from number theory. He called it the *aureum theorema,* "the golden theorem", and he managed to devise as many as eight different proofs of it over his lifetime. [21]

If p and q are distinct odd primes, then the congruences

$x^2 \equiv q \pmod{p}$

$x^2 \equiv p \pmod{q}$

are both solvable or both unsolvable unless both p and q leave the remainder 3 when divided by 4 (in which case one of the congruences is solvable and the other is not).

Written symbolically this reads:

Let p and q be odd primes, then

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{(p-1)(q-1)/4}$$

where the Legendre symbol $\left(\frac{p}{q}\right) \equiv 1$ for $x^2 \equiv p \pmod{q}$ solvable for x

$\left(\frac{p}{q}\right) \equiv -1$ *for* $x^2 \equiv p \pmod{q}$ not solvable for x

This was a significant achievement considering that it enables us to solve almost all quadratic congruences. It is certainly not an easy theorem to prove, causing difficulty for both Euler and Legendre. This is another example of how important primes are and their fundamental role in the development of number theory. [22]

At first sight the primes seem to be distributed among the integers in rather a haphazard way. For example in the 100 numbers immediately before 10000000 there are 9 primes, while in the 100 numbers after there are only 2 primes. However, on a large scale, the way in which the primes are distributed is very regular. Legendre and Gauss both did extensive calculations of the density of primes. This search for the order of magnitude of two arithmetical functions related to primes led to one of the most profound theorems in number theory, the prime number theorem. Both came to the conclusion that for large n the density of primes near n is about $\frac{1}{\log(n)}$.

They came to their conjectures independent of each other — neither were able to provide proof.

Legendre gave an estimate for $\pi(n)$ the number of primes $\leq n$ of

$$\pi(n) = \frac{n}{(\log(n) - 1.08366)}$$

While Gauss gave his estimate in terms of the logarithmic integral

$$\pi(n) = \int_2^n \frac{1}{\log(t)}\,dt$$

Gauss was led to his conjecture by examining a table of primes $\leq 10^6$. He calculated $\pi(n)$ simply by counting the number of primes in the interval from 1 to n. Gauss, Legendre and many other mathematicians of the early 19th century tried unsuccessfully to prove this conjecture. [23]

The Prime Number Theorem

The prime number theorem eluded mathematicians for years:

$$\lim_{n \to \infty} \frac{\pi(n)}{n/\log n} = 1$$

The first step towards a proof was made in 1851 by Chebyshev, who showed that if $\frac{\pi(n)\log n}{n}$ has a limit as $n \to \infty$, then this limit must equal 1. In 1859 Riemann attacked the problem with a new method, using a formula of Euler's relating the sum of the reciprocals of the powers of the positive integers with an infinite product extended over the primes. Emerging with what is now called the Riemann zeta function, he did much relating to the distribution of the prime numbers. Riemann came close to proving the prime number theorem, but not enough was known during his lifetime about the theory of functions of a complex variable to complete the proof successfully. Thirty years later the necessary analytic tools were at hand, and in 1896 Hadamard and Vallee-Poussin independently proved the prime number theorem. [24] The proof was one of the great achievements of analytic number theory. Subsequently new proofs were discovered, including an elementary proof found in 1949 by Erdos and Selberg that makes no use of complex function theory.

The prime number theorem is important not only because it makes an elegant and simple statement about primes and has many applications but also because many new mathematical discoveries were made in the attempts to find a proof. [25]

In 1876, French mathematician Edouard Lucas developed a method for testing the primality of Mersenne primes. He used it to prove that $2^{127}-1$ is prime. To this day it stands as the largest prime found by hand calculations. Around 1951, an American mathematician named Dick Lehmer refined Lucas' method and programmed it into one of the first computers called SWAC. On January 30, 1952, he used the computer to prove that $2^{521}-1$ is prime. On the same day, he also had the opportunity to prove that $2^{607}-1$ is also prime. The SWAC team went on to find two more Mersenne primes that year, but larger primes seemed to be beyond the reach of that machine.

Since then it has almost become traditional to christen the latest supercomputer by giving it a record high prime to prove. Not only does it give the hardware a good workout, it gives the computer manufacturer some kudos! All that changed in 1995, as the first of 5 Mersenne primes were discovered using desktop computers. Using free Lucas-Lehmer software and coordinated over the Internet, the Great Internet Mersenne Prime Search (GIMPS) has tested systematically every possible Mersenne number for primality. [26]

Searching for Primes

As you might suspect, these records are not shattered using the Sieve of Eratosthenes, they used more complicated but much faster methods. For example, to prove a number n is prime, you do not really need to test if any of the numbers below it are factors. All you need to test are if any primes less than the square root of n are factors. This lessens the calculating load considerably, as long as you know all the primes less than the square root of n. Over the years, mathematicians and programmers have discovered other shortcuts, and have written software to test for primes that is faster and more efficient with each generation.

For the past century, programmers have searched relentlessly for a prime routine that would test for primes of complexity P. In order for a routine to be of complexity P, it has to have two qualities. Firstly, the routine must be deterministic which means it always returns the exact right answer. It is much easier to find routines that are non-deterministic, which are right 99.99999% of the time, good enough for most uses but we can never be sure they are exactly right. Such routines are called NP class. The complexity must be a polynomial, that is to say as n increases the time it takes to come up with an answer is less than n^c where c is a constant. No variables are allowed in the exponent.

The need for the complexity to be a polynomial is evident when considering growth. For example it is helpful to compare how quickly 2^n rises as n gets bigger with the growth of n^2.

2^n gives 2, 4, 8, 16, 32, 64, 128, 256, …

n^2 gives 1, 4, 9, 16, 25, 36, 49, 64

In the beginning, they grow at about the same rate, but by the time n=6, 2^n is already nearly twice as big, and the difference increases even further from there. Until recently, every test for primality has either been non-deterministic or of exponential complexity. Mathematically, complexity is represented as O(f(n)), where f(n) is the growth in time it takes to get an answer as n increases. Complexity always takes into consideration the worst-case scenario. In 1980, Adleman and Rumely developed a test that would decide if a randomly chosen number up to 100 digits was prime in 4 to 12 hours with a large computer, such as a CRAY supercomputer. Around 1983, Cohen and Lenstra improved the program around 1000 times, so now a 100-digit number could be tested for primality in about 40 seconds on a supercomputer. The routine they used had a complexity of $(\log n)^{O(\log\log\log n)}$ which is considerably better than exponential time, but since n is still in the exponent, it is still not polynomial time and so not in P. Since then there have been programs based on prime conjectures which would run in polynomial time, but since the conjectures were unproven, they were not valid. [27]

While the goal of a P test for primes remained elusive, in 1980 Michael Rabin invented a "monte carlo" test in P that would quickly determine if a very large number was likely to be a prime. To decide if number p was prime, Rabin would pick a random number r between 2 and p – 1 and apply two tests:

$$r^{p-1} > 1(\text{mod } p)$$

$$\textit{and for some } k, 1 < \gcd(r^{(p-1)/2k} - 1, p) < p$$

The first test is based on Fermat's little theorem and the second test is obvious since if p has a factor>1 it is not prime by definition. By the theorem, better than half of the r values between 2 and p-1 will pass both tests if p is not prime, thus according to Rabin they are "witnesses to the compositeness of p". So, if both tests pass, then p is not prime. If a test fails then there is better than 50 % chance that p is prime. So we pick a second r and try the tests again. If another test fails, then the odds of p being prime is now 75%. For every r that we pick and the test fails, we double the likelihood of p being prime. So picking 10 distinct r and all of them failing leads to a 99.9% likelihood of p being prime, which is pretty good. Other "Monte Carlo" prime tests have been proposed, but as accurate as the tests are, they still do not guarantee the right answer, thus they are "non deterministic" and not in P.

In 2002, three computer scientists at the Indian Institute of Technology in Kanpur named Agrawal, Kayal and Saxena devised a new "Monte Carlo" test based on a corollary of Fermat's little theorem and found a small set of r that would determine if a number is prime.

Suppose that a and p are relatively prime integers with p>1. p is prime if and only if

$$(x-a)^p \equiv (x^p - a)(\text{mod } p)$$

$(x-a)^p$ results in a polynomial of degree p with p+1 terms. All but the first and the last term are going to be positive or negative integers with absolute value >1. If p is prime and a is relatively prime to p, then all the co-efficients will be divisible by p, thus modulo p of each term =0 except for the first term x^p and the last term $-a^p$, which modulo p= -a, since a and p are relatively prime. The problem that remains is that it still takes exponential time to determine if p is prime using this test because you have to calculate p-1 coefficients and divide all by p to see if there is a remainder. As yet an actual program to apply the test has yet to be written. It has a large complexity, but most importantly it is polynomial but it may be too much of a memory burden to test on really big numbers. The importance of this discovery is that finding primes is indeed possible in P, making much easier routines theoretically possible. [28]

Primes and Codes

Primes relate to ideas that have important practical applications. Increasingly, modern society depends for its operation on fast and accurate information. Sometimes it is desirable to keep information secret from everyone except the sender and recipient. Examples of such information are diplomatic communications, money transfers between banks, medical records and business transactions. Secret codes and ciphers come into play in such situations. The science of cryptography, which deals with the making and breaking of codes, involves specialists from linguists to engineers, mathematicians and computer scientists. The widespread use of computers and telecommunications equipment has made necessary more flexible codes and ciphers. In 1977, Rivest, Shamir and Adleman made it feasible to implement new public key systems. They had a clever idea to exploit the contrast between the speed of primality testing and the apparent difficulty of factoring. The design of the system depends on being able to choose very large integers that are known to be prime. Thus, the issue of checking whether or not a large number is prime becomes significant in this system. [29]

The RSA cryptosystem

Choose two large prime numbers p, q of about 100 digits each

Let n = pq. Let the plaintext be broken into a sequence of positive integers m < n.

$\phi(n) = (p-1)(q-1)$

Choose a random number e relatively prime to $\phi(n)$ and carry out the encryption function

$c \equiv m^e \bmod n$ to obtain the corresponding cipher text c.

As gcd $(e, \phi(n)) = 1$

We may find an integer less than n such that $ed \equiv 1 \bmod \phi(n)$

Then we can decrypt c by calculating m $\equiv c^d \bmod n$

The mathematics that underpins the RSA cryptosystem is derived from the following result that depends on the findings of Euler's theorem.

Let p,q be distinct primes, n = pq and let e, d be positive integers such that ed $\equiv 1 \bmod \phi(n)$. If 0 $\leq m < n$ *and* $m^e \equiv c \bmod n, r \equiv c^d \bmod n$ where 0 $\leq r < n$ then r = m.

The new algorithm that verifies that testing for primality can be done in polynomial time is not speedy enough to be used in crypto graphical applications unless it can be modified in some way that speeds it up significantly. Furthermore, the fact that numbers can be tested in polynomial time to see if they are prime in no way affects the security of RSA to the best of current knowledge. RSA'*s* security depends on the complexity not of checking if a number is prime but of factoring numbers into primes. [30]

We have now seen how primes have crucial practical applications in today's world by providing secure networks. An increasing reliance on computer data banks for anything from medical histories to credit records has ensured that primes and their place in cryptography have become indispensable. We have also seen the ongoing search into finding the largest primes and the largest primes of a certain kind such as Mersenne primes and Fermat primes. The other area of activity, that has not yet been discussed, concerns the unsolved problems surrounding primes.

Unsolved Problems

We saw earlier how much interest surrounds Goldbach's conjecture. There is an easier case of Goldbach's conjecture. It is known as the odd Goldbach problem: Every odd n > 5 is the sum of three primes. There has been substantial progress made on this problem. In 1937 Vinogradov proved that this is true for sufficiently large odd integers n. In 1956 Borodzkin showed $n > 3^{14348907}$ is sufficient. In 1989 Chen and Wang reduced this bound to 10^{43000}. While this has significantly reduced the nature of the problem, the exponent still must be reduced dramatically before computers can be used to take care of all the same cases. [31]

Another conjecture that attracts a lot of interest concerns twin primes. Twin primes are pairs of primes of the form (p, p+2). Paul Stackel at the start of the 20th century coined the term 'twin prime'. It can be observed that there are many twin primes to be found among the first 100 positive integers such as (17, 19), (29, 31), (41, 43) for example. It is conjectured that there are an infinite number of twin primes, but proving this remains one of the most elusive open problems in number theory. An important result for twin primes is Brun's theorem, found in 1919, which states that the number obtained by adding the reciprocals of the odd twin primes

$$B \equiv (\frac{1}{3}+\frac{1}{5})+(\frac{1}{5}+\frac{1}{7})+...$$

converges to a definite number known as Brun's constant B = 1.902160577… This expresses the scarcity of twin primes, even if there are infinitely many of them. [32]

In 1849, Polignac conjectured that for every even number 2n, there are infinitely many pairs of consecutive primes which differ by 2n. When n = 1 this is the twin prime conjecture. Another example is when n=2, the even number is 4 and 7,11 would be the first pair to satisfy the condition of consecutive primes differing by 4.13, 17 would be the next pair and so on. Whether there is an infinite number of such pairs remains an open question. [33]

In 1882, Opperman stated $\pi(n^2+n) > \pi(n^2) > \pi(n^2-n)$ (n > 1), which seems likely but remains unproven. A more general statement of this asks is there always a prime between n^2 and $(n+1)^2$. Both of these conjectures would follow if we could prove the conjecture that the prime gap following a prime p is bounded by a constant times $(\log p)^2$. Similarly Bertrand conjectured that for every positive integer n > 1, there exists at least one prime p satisfying n < p < 2n. Unlike the previous two conjectures though, this result was proven in 1850 by

Chebyshev. [34]

Are there infinitely many primes of the form n^2+1? Another straightforward question but again one that cannot be answered. Dirichlet proved that given an arithmetic series of terms an + b, for n = 1, 2, …, the series contains an infinite number of primes if a and b are relatively prime. Dirichlet proved this theorem using Dirichlet L-series. [35] It is interesting to see the diverse methods that are used by mathematicians to approach questions that have eluded people for years.

There are many more unsolved problems in the theory of numbers relating to primes. To just name some of them, they include:
Are there infinitely many primes of the form n# + 1? (Where n# is the product of all primes ≤ 1)
Are there infinitely many primes of the form n# -1?
Are there infinitely many primes of the form n! + 1?
Are there infinitely many primes of the form n! –1?
If p is a prime, is 2^p-1 always square free? (Meaning that it is not divisible by the square of a prime.)
Does the Fibonacci sequence contain an infinite number of primes?

There are many open questions of this type. As discussed earlier, the Goldbach conjecture is probably the most famous of these. These open questions are fascinating since they do not require a big amount of background knowledge. There is very little to be learnt about number theory in order to comprehend what a conjecture refers to, this is in contrast to a problem such as the Riemann hypothesis. [36] Such conjectures are very beneficial to the growth of mathematics as a science. Very often mathematicians come upon new results and findings and extend knowledge when searching for solutions to these kinds of open problems. Students can easily tread the same steps as mathematicians and see the process that they patiently persist with.

Conclusion

Our journey has come to an end; the importance of primes and how they achieved their place of importance within mathematics has now been charted. Primes have been studied since the time of the Pythagoreans and this study continues today. Some of the most beautiful mathematics that has been discovered and proved began with the study of the primes. With the coming of the age of the computer, primes achieved a practical use and are now essential given the need for encryption in today's world. Primes and their problems are relatively simple to understand but many of the questions involving primes have shown themselves to be extremely difficult to answer.

References

1. Math Lair, *Prime Numbers* http:// www.stormloader.com/ajy/primes.html
2. J. J. O'Connor and E. F. Robertson, *Primes* http://www.gap.dcs.st-and.ac.uk/history/HistTopics/Prime-numbers.html
3. C. Caldwell *Prime FAQ* http://www.utm.edu/research/primes/notes/faq/one.html
4. D.M. Burton, *Elementary Number Theory*, 1976, p.53
5. Jones & Jones, *Elementary Number Theory*, 1988, p.26
6. D. J. Struik, *A Concise History of Mathematics*, 1987, p.101
7. *Primes is in P* http://members.cox.net/mathsmistakes/primes.htm
8. G. E. Andrews, *Number Theory*, 1971, p.49
9. W. Sierpinski, *Elementary Theory of Numbers,* 1988, p.131
10. C. Caldwell, *Euler's Phi Function*, http://primes.utm.edu/glossary/page.php?sort=Eulersphi
11. C. Caldwell, *Euler's Theorem*, http://primes.utm.edu/glossary/page.php?sort=EulersTheorem
12. C. Caldwell, *Goldbach's Conjecture*, http://primes.utm.edu/glossary/page.php?sort=Goldbachconjecture
13. *Goldbach's conjecture* http://en.wikipedia.org/wiki/Goldbach's_conjecture
14. Math World *Goldbach Conjecture* http://mathworld.wolfram.com/GoldbachConjecture.html
15. *Wilson's Theorem* http://en.wikipedia.org/wiki/Wilson's_theorem
16. C. Caldwell, *Wilson's Theorem,* http://primes.utm.edu/glossary/page.php? Sort=WilsonsTheorem
17. Math World *Wilson's Theorem* http://mathworld.wolfram.com/WilsonsTheorem.html
18. *Wilson Prime* http://mathworld.wolfram.com/WilsonPrime.html
19. C. Caldwell, *Wilson Prime* http://primes.utm.edu/glossary/page.php?sort=WilsonPrime
20. Foldoc *Carl Friedrich Gauss* http://wombat.doc.ic.ac.uk/foldoc/foldoc.cgi?Carl+Friedrich+Gauss
21. *The Mathematics of Carl Friedrich Gauss* http://members.fortunecity.com/kokhuiton/gauss.html
22. Math World, *Quadratic Reciprocity Theorem* http://mathworld.wolfram.com/quadraticreciprocitytheorem.html
23. Math World, *Prime Number Theorem,* http://mathworld.wolfram.com/primenumbertheorem.html
24. Foldoc *Prime Number Theorem* http://wombat.doc.ic.ac.uk/foldoc/foldoc.cgi?Prime+Number+Theorem
25. *The Prime Number Theorem* http://users.torthnet.gr/ath/kimon/pnt/prime%20number%20theorem.htm
26. D. Surendran *Prime Numbers* http://www.geocities.com/capecanaveral/lab/3550/prime.htm
27. *Primes is in P* http://members.cox.net/mathsmistakes/primes.htm
28. *Primes – The Right Stuff* http://www.ams.org/new_in_math/cover/primes6.html
29. S. Landau *Primes, codes and the National Security Agency* http://www.totse.com/en/privacy/encryption/primes.html
30. *Primes and Cryptography* http://www.ams.org/new_in_math/cover/primes5.html
31. C. Caldwell *Prime conjectures and open questions* http://www.utm.edu/research/primes/notes/conjectures
32. Math World *Twin Primes* http://mathworld.wolfram.com/TwinPrimes.html
33. *Prime numbers* http://students.bath.ac.uk/ch1au/prime%20numbers.html
34. *Bertrand's conjecture* http://planetmath.org/encyclopedia/bertrandsconjecture.html
35. Math World *Dirichlet's Theorem* http://mathworld.wolfram.com/DirichletsTheorem.html
36. C. Caldwell *The Riemann Hypothesis* http://www.utm.edu/research/primes/notes/rh.html

Customizing Toolbars for Maths Teachers

Microsoft Word is not just for letter writing. Word is an extremely powerful and helpful program that can be used in a variety of ways to enhance learning and teaching and to make a teacher's job much easier.

Microsoft Office products use Toolbars to provide shortcuts to menu commands. These Toolbars are generally located just below the Menu bar. The two standard toolbars are **Standard** and **Formatting**.

The Standard Toolbar

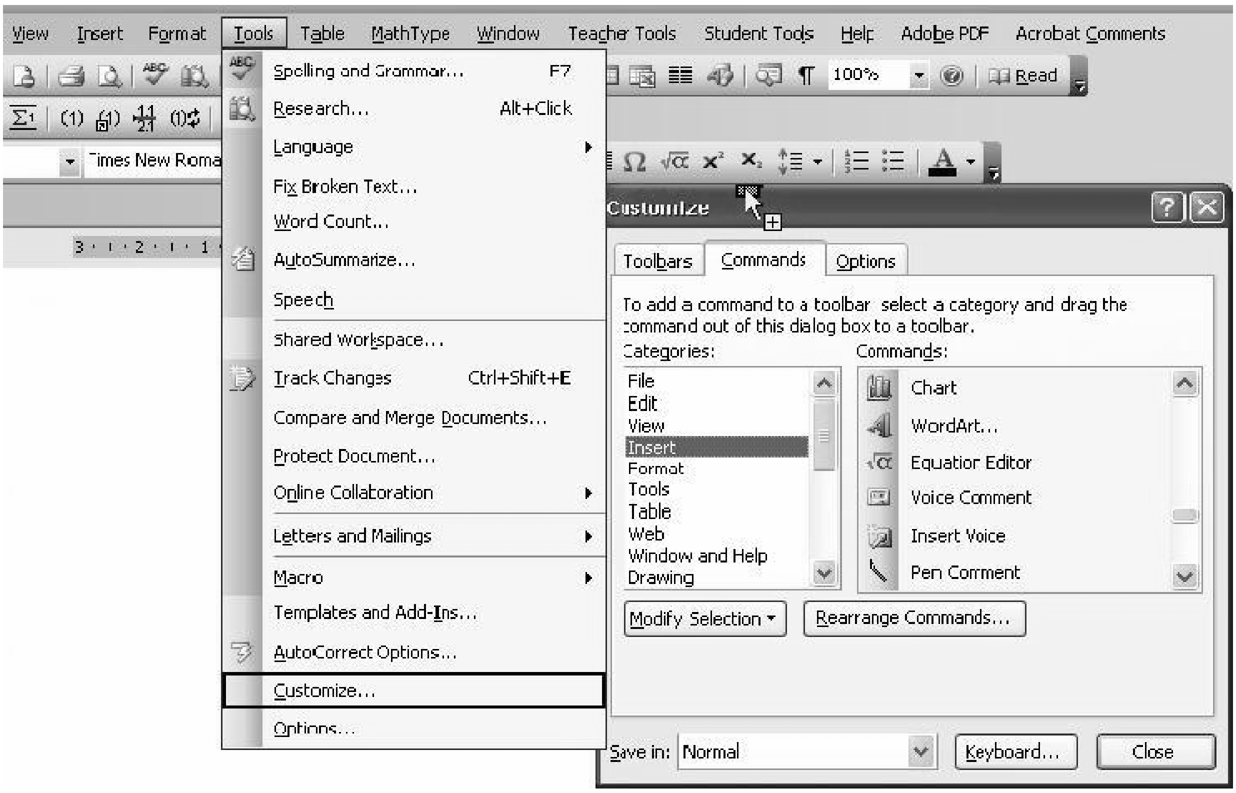
The Formatting Toolbar

1. Click View on the Menu bar.
2. Highlight Toolbars.
3. Standard and Formatting should have check marks next to them. If both Standard and Formatting have check marks next to them, press Esc three times to close the menu.
4. If they do not both have check marks, click Customize.
5. Click the Toolbars tab.
6. Point to the box next to the unchecked option and click the left mouse button to make a check mark appear. Note: You turn the check mark on and off by clicking the left mouse button.
7. Click Close to close the dialog box.

These Toolbars are designed for standard word processing, but there are a lot of hidden buttons that can be added to make it so much easier to prepare Maths worksheets and exam questions. There are 4 key buttons which will make life much easier when writing Maths equations – Symbols, Superscript, Subscript and Equation Equator.

How to Add a Button 

1. In the **Tools** menu, click on **Customize**.
2. Click the **Commands** tab. Under **Categories** select the category from which you want to add a button.
3. The available built-in buttons for the selected category are displayed to the right of the list (under **Commands**).
4. Click the button you want, and drag it to the location you want on a toolbar.
5. Symbols and Equation Editor are in the **Insert** category while Superscript and Subscript are in the **Format** category.



Equation Editor is brilliant if you want to add $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ to a document.

Every symbol you could ever wish for is available and it is great when you're doing heavy algebra or calculus. It does however require you to open a dialog box, do your equation and then close the box when you're finished. It's a bit of a sledgehammer when you want to just type $3x^2 + 2x - 5$.

The Superscript button, once you have it installed is much easier and quicker to use. To do this just type 3x2 + 2x – 5, highlight the x with your mouse and click the **Italic** button, then highlight the 2 and click the Superscript button and presto you have $3x^2 + 2x - 5$. Way easier than doing this in Equation Editor.

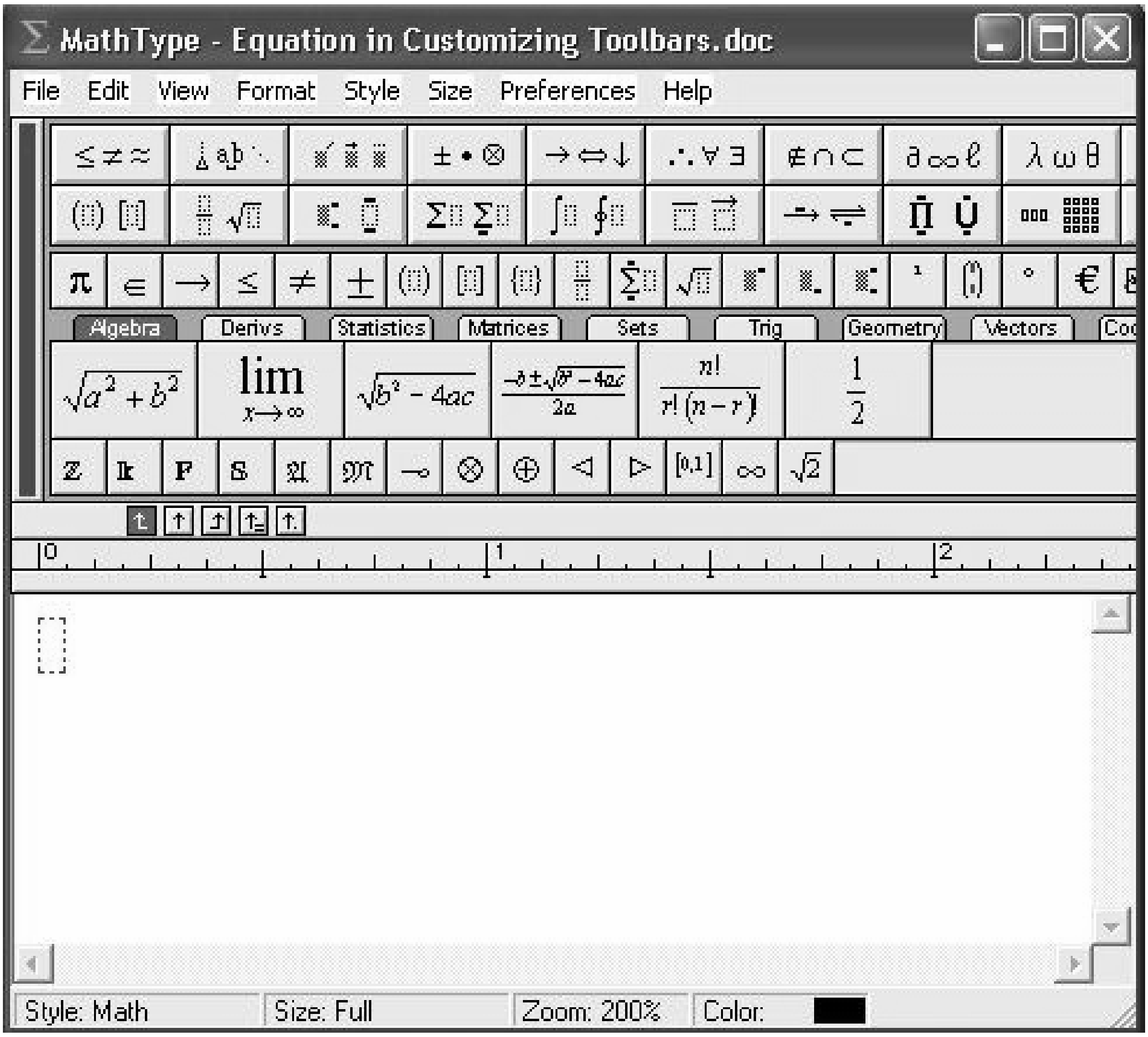
The Subscript button is very useful for Co–ordinate Geometry (x_1, y_1) and Series and Sequences etc and again far easier to use than Equation Editor for this purpose.

The Symbol button is a great help when you just want a one–off symbol rather than a full equation. $\angle$ ° > Δ × ÷ ∩ ∪ ⊂ π ± ∈ ⊥ ≤ ≥ ≠ | and lots more, are there at the click of a button.

Finally, a wonderful upgrade version is available for Equation Editor. **Maths Type 5.2** is available as a 30 day free trial download from a company called Design Science (the original developers of Equation Editor) at their website:
http://www.dessci.com/en/products/mathtype/trial.asp

Maths Type 5.2 has a lot of extra features such as colour equations and edit copy/paste with a mouse button click, customizable equation toolbars and so on. All very nice if you do a lot of computer work, but the biggest advantage of Maths Type is editing equations in a separate window, as shown in the above graphic, rather than "in-place" which is standard with Equation Editor. The viewing scale of the equation windows can be controlled independently of the viewing scale of the word processing or presentation window. Whereas you might want to edit your document at a scale of 100% or 125%, equations are sometimes hard to read and edit at such scales. 200% scale in your equation windows is recommended.

The great news is that after the 30 day trial Maths Type goes into **Lite Mode** and many of the extra features such as edit copy/paste with a mouse button click, and the separate window for editing equations remain and you've saved yourself $US99.95!!!



In a future Newsletter I will outline some tips and ideas for Microsoft Powerpoint. If you are in the Dublin area, I am giving evening classes on Maths and ICT at Blackrock Education Centre and Dublin West Education Centre.
To book, contact Blackrock Education Centre: Phone 01 2365000, Fax: 01 2365050
Email: bec@blackrockec.ie Web: www.blackrockec.ie

Dublin West Education Centre: Phone 01 4528000, Fax: 01 4528010
Email:info@dwec.ie Web: www.dwec.ie
Both courses are free of charge.
If you are interested in an Interactive CD for Leaving Cert Maths, you can contact me on 085 7305 963.

Horst Punzet
Dublin

The Dame and the Knights of the Square Table

or how to construct magic squares without the sums

Magic Squares evoke memories of 40 minute classes which seemed to last endless hours, usually punishingly relentless, monotonous hours. The classic class stocking filler of old has been long overdue an overhaul, and the February issue of the Institute of Mathematics and its Applications' (IMA) newsletter features an article by the amazing 93 year old Dame Kathleen Ollerenshaw, who injects new tricks into this old magic. Declaring her *Constructing PanDiagonal Magic Squares of arbitrarily large size* article as her definite final swansong, the grand old Dame continues to defy conservative wisdom regarding age and sex in Mathematics to produce clear and incisive Mathematics.

10	4	8	22	**16**
7	21	15	14	3
19	**13**	2	6	20
1	5	**24**	18	12
23	17	11	**0**	9

The 5x5 magic square opposite is so called because all rows and columns add to 60, its magic constant. Furthermore it is called a Pandiagonal Magic Square (PDMS), because both principal diagonals sum to the magic constant and the broken diagonals sum to the magic constant, one of which is shown in bold. There are many more patterns in the square which can sum to the magic constant. Also, for an nxn square, the value of the constant is $n(n^2-1)/2$, if the numbers 0 to n-1 are used. Dame Ollerenshaw's article describes how to construct any nxn magic square, where n is a prime number greater than 3, using a base n and an adaptation of a Knight's move in Chess. In this article I describe how the technique for constructing a 5x5 square was taught to a group of Higher Level Transition Years over a period of 3 classes. The students enjoyed it, and I certainly enjoyed it. Here's a summary of how it was taught.

Have a Go at Making A Magic Square

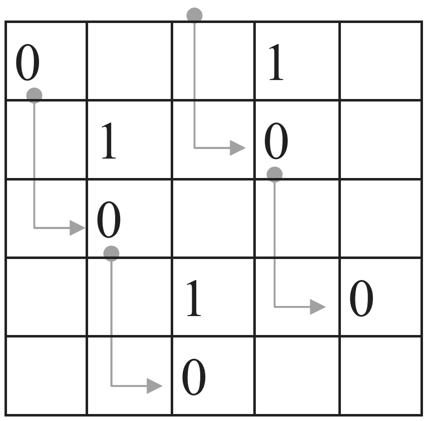
I began by asking the students to compose a 5x5 magic square using the numbers from 0 to 24 inclusive, like the one given above. It is surprising which students take a shine to the task. Since your average whizzkid would find the task too difficult, I scaled the problem down to a 3x3 magic square using the numbers 1 to 9 inclusive, and this got most of them going.

Numbers of a Different Base

With Logs removed from the Junior Cycle but lying in wait for students in Higher Level Senior Cycle, the idea of numbers in a different base is foreign to most students but helpful for them if they are exposed before beginning 5th year. Learning bases in the context of constructing these squares at least gives an immediate purpose to learning them. In class, I recapped that 347 in base 10 is made up of $7 \times 10^0 + 4 \times 10^1 + 3 \times 10^2$. I then put up a completely different number, 347, at which point they thought all mathematical sense had deserted me. But what if the base is different? What is 347 in base 8 or base 9? Lots of examples in a variety of bases were necessary, particularly base 5. It is important that they are comfortable changing numbers from base 10 to base 5 and vice versa, especially the numbers from zero to twenty four. Once this is done, the construction steps can be followed as described below, which is more or less how Dame Ollerenshaw describes it.

Constructing the PanDiagonal Magic Square

A 5x5 square is drawn which will be populated by five 0's, five 1's, five 2's, five 3's and five 4's.



0 Place the first zero anywhere. For explanatory purposes, we'll place it in row 1, column 1 (R1C1).

- Using the Knight's move from Chess, the square for the next zero is found by going **down two squares and right one square**, as shown in the diagram opposite. This brings us to R3C2, as shown.
- Using the same Knight's move, brings us to the square at position R5C3.
- This third zero has no squares down below it, so the move will wrap around to the top, come down two squares in column 3, and across one square to the next zero in position R2C4.
- The final zero is then located as shown on the diagram, at R4C5.

1 Any vacant square will do, but again we'll continue along the leading diagonal, and place the first one in square R2C2.

Following the same Knight's tour, brings us to R4C3, then some wrapping brings us to R1C4, then R3C5, and wrapping around row 5 brings us to R5C1.

234 Continuing in this manner gives the completed square below labelled "5^1 digits" (Technically it's known as the Radix Square).

The "5^0 digits" square is constructed in the same fashion as the above but the Knight's movement for each digit is **down 1 square and right 2 squares**, as can be seen. It is technically known as the Unit Auxiliary Square. Again the first digit can be placed anywhere, but for illustrative purposes the leading diagonal is composed of 01234, each one of these being the first digits in the construction. These two squares are said to orthogonal, since one is the image of the other under an axial symmetry in the leading diagonal.

0	2	4	1	3
4	1	3	0	2
3	0	2	4	1
2	4	1	3	0
1	3	0	2	4

5^1 digits

0	4	3	2	1
2	1	0	4	3
4	3	2	1	0
1	0	4	3	2
3	2	1	0	4

5^0 digits

Bringing it all Together

The final magic square is easily constructed in base 5 by combining correspondingly placed digits to form the square below labelled PDMS Base 5. The students, well versed at this stage in jumping bases, can easily convert to PDMS base 10. When it's done there's a sense of a bit of black magic being cast over the whole process, especially if it all started with students trying to create a 5x5 square from scratch. The magic square given at the beginning was constructed using randomly placed first digits, each successive digit following the knight's tour.

This can be easily unbundled back to the original radix and unit auxiliary squares. This technique works for all nxn squares, where n is a prime greater than 3. In her article, the great Dame describes more intricate techniques for constructing squares of any given dimension, including a 9x9 square which follows Su Doku principles.

00	24	43	12	31
42	11	30	04	23
34	03	22	41	10
21	40	14	33	02
13	32	01	20	44

PDMS Base 5

0	14	23	7	16
22	6	15	4	13
19	3	12	21	5
11	20	9	18	2
8	17	1	10	24

PDMS Base 10

Paul Behan
Kilkenny

Comórtas Sóisearach Matamaitice Éireann 2007
(Irish Junior Mathematics Competition 2007)

This competition is organised by the Irish Mathematics Teachers Association (I.M.T.A.)

Eligibility First Year Students 2006/07

Format One set of question papers and answer key will be posted to each participating school some days before the competition date.

Each school will be responsible for photocopying the question paper and administering the First Round.

First Round Wednesday, March 14th, 2007 Time : 40 minutes

Final May 2007

The top students from the First Round may be invited to compete in the Final at venues to be arranged, ***provided a certain standard is reached.***

Entrance fee €20 **per school** (cheques payable to I.M.T.A)

If you wish your school to participate please return the completed Registration Form with the fee
no later than November 30th 2006

Applications received after this date may not be accepted.

Applications should be sent to:

Michael D. Moynihan (Mícheál D. Ó Muimhneacháin)
Coláiste an Spioraid Naoimh,
Bishopstown,
Cork.

Phone : 087-2860666 / 021- 4870362 (Evenings)
email mmoynihan@eircom.net

Fax : 021 - 4543625

Registration Form 2006

Name of Teacher:____

School:____

School address____

Phone number________________(Home)__________________________(School)

School Fax number_____________________ email address___________________

Approximate number of students participating_______________________________

Estimating the Cost of Filling the 2006 GAA Players Sticker Album using Euler's Approximation for Harmonic Sums

In order to complete the official 2006 GAA players sticker album, fans must collect 376 different stickers. Stickers are sold in packets of 8 at a cost of €0.99 per packet. The packets contain a random selection of stickers and may have multiple copies of the same sticker. One would like to know what is the expected average cost to fill one album? More interestingly, how did the famous mathematician Leonhard Euler (1707-1783) work out the answer over 100 years before the GAA was founded?

The Expected Average Cost

Ignore (initially anyway) the fact that stickers come in packets of 8**.** Instead, consider examining stickers one at a time. The first sticker examined will be new to our album with a probability of 1, denoted by $p_1 = 1$. The waiting time T_1 (measured in number of stickers examined) until this event takes place satisfies $p_1 \times T_1 = 1$ and so $T_1 = 1$. After collecting our first sticker, there are still 375 stickers (from a possible 376) to collect. The probability of finding a second sticker suitable for inclusion in our album is $p_2 = 375/376$. The average waiting time T_2 until this event takes place satisfies $p_2 \times T_2 = 1$ and so $T_2 = 376/375$, that is we expect to have to inspect an average of $T_2 = 376/375$ stickers in order to observe a new sticker for inclusion in our album. The same argument can be used to calculate the average waiting times until the third, fourth and fifth stickers are found, so that $T_3 = 376/374$, $T_4 = 376/373$, $T_5 = 376/372$, and so on. The average waiting time (measured in number of stickers required) to complete the album is

$$376\left(\frac{1}{376}+\frac{1}{375}+\cdots+\frac{1}{2}+\frac{1}{1}\right)=2447.058 \quad . \qquad (1)$$

Since the stickers come in packets of 8, the average number of packets required to fill the album is 2447.058/8 = 305.88, or rounding up we get 306. At a cost of €0.99 a packet this implies a total expected average cost of €303 per album.

Euler's Approximation For Harmonic Sums

The calculation in (1) is the key step to obtaining the answer of our sticker collection problem. Nowadays, the task of calculating and summing the reciprocals in equation (1) is straightforward using a computer calculator programme or a spreadsheet. When the famous mathematician Leonhard Euler (1707-1783) needed to complete a calculation similar to (1) he came up with the following ingenious approximation to the partial sum of the harmonic series:

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\approx \log_e n+\frac{1}{2n}+0.57721 \,, \qquad (2)$$

where 0.57721 is now known as Euler's constant. For an album with n stickers, Euler's approximation of the average waiting time to complete the album is obtained by multiplying equation (2) by n, giving $n\log_e n+0.57721n+1/2$. For n =376, Euler's approximation yields $(376\times\log_e 376)+(0.57721\times 376)+1/2=2447.056$, very close to the true answer 2447.058. Euler's constant, often denoted by the lower-case Greek letter gamma γ, plays an important role in number theory.

The Coupon Collector's Problem

Filling the GAA players stickers album is an example of what is known in mathematics as the coupon collector's problem. Suppose that independent trials, each of which results in any of n possible outcomes (number of football stickers to collect for our album) with probability n^{-1}, are continually performed. For the GAA players sticker album n = 376 . Let W denote the number of trials needed (stickers examined) until each outcome has occurred at least once. For $w\geq n$, the rules of probability (see Dawkins 1991, for details) imply

$$\Pr(W\leq w)\approx \exp\left[-n\exp\left(-\frac{w}{n}\right)\right]$$

Suppose we want to find the number of packets such that 50% of collectors will need fewer than this number and 50% will need more than this number to complete their album. We need to solve for w in the formula above such that Pr($W\leq w$) = 0.50. Using the formula we get w = 2368 stickers or w = 296 packets of 8 stickers. Hence

$$\Pr(W\leq 296\times 8)\approx \exp\left[-376\exp\left(-\frac{296\times 8}{376}\right)\right]=0.50 \;.$$

This means that 50% of collectors would be expected to need fewer than 296 packets in order to complete the album, and 50% of collectors would be expected to need more than 296 packets to complete the album. The value 296 is the median value as it splits the population of collectors in two equal halves. The cost of 296 packets is just over €293.

It is well known that the average (or mean) value need not split a sample in two equal parts. For example, use the formula with $w = 306$ packets of 8 stickers where 306 is the average number of packets required to complete the album (calculated earlier).

$$\Pr(W \le 306 \times 8) \approx \exp\left[-376 \exp\left(-\frac{306 \times 8}{376}\right)\right] = 0.55 \ .$$

This means that 55% of collectors would be expected to need less than the average number of 306 packets in order to complete the album (i.e., spend just less than €303) and 45% of collectors would be expected to need more than that amount.

We can also use the formula to calculate

$$\Pr(W \le 264 \times 8) \approx \exp\left[-376 \exp\left(-\frac{264 \times 8}{376}\right)\right] = 0.253 \ .$$

This means that a lucky 25% of collectors would be expected to need less than 264 packets in order to complete the album (i.e., spend less than €261.36). However, we can also use the formula to calculate

$$\Pr(W \le 337 \times 8) \approx \exp\left[-376 \exp\left(-\frac{337 \times 8}{376}\right)\right] = 0.75 \ .$$

This means that an unlucky 25% of collectors will be expected to collect more than 337 packets to complete their album (estimated cost greater than €333.63).

A computer simulation approach gives approximately the same answers as these calculations. The number of packets required in order to complete 10,000 simulated sticker albums were calculated using the software package R (R Development Core Team, 2003). The simulation results (empirical) along with theoretical values (calculated earlier) are displayed in Figure 1.

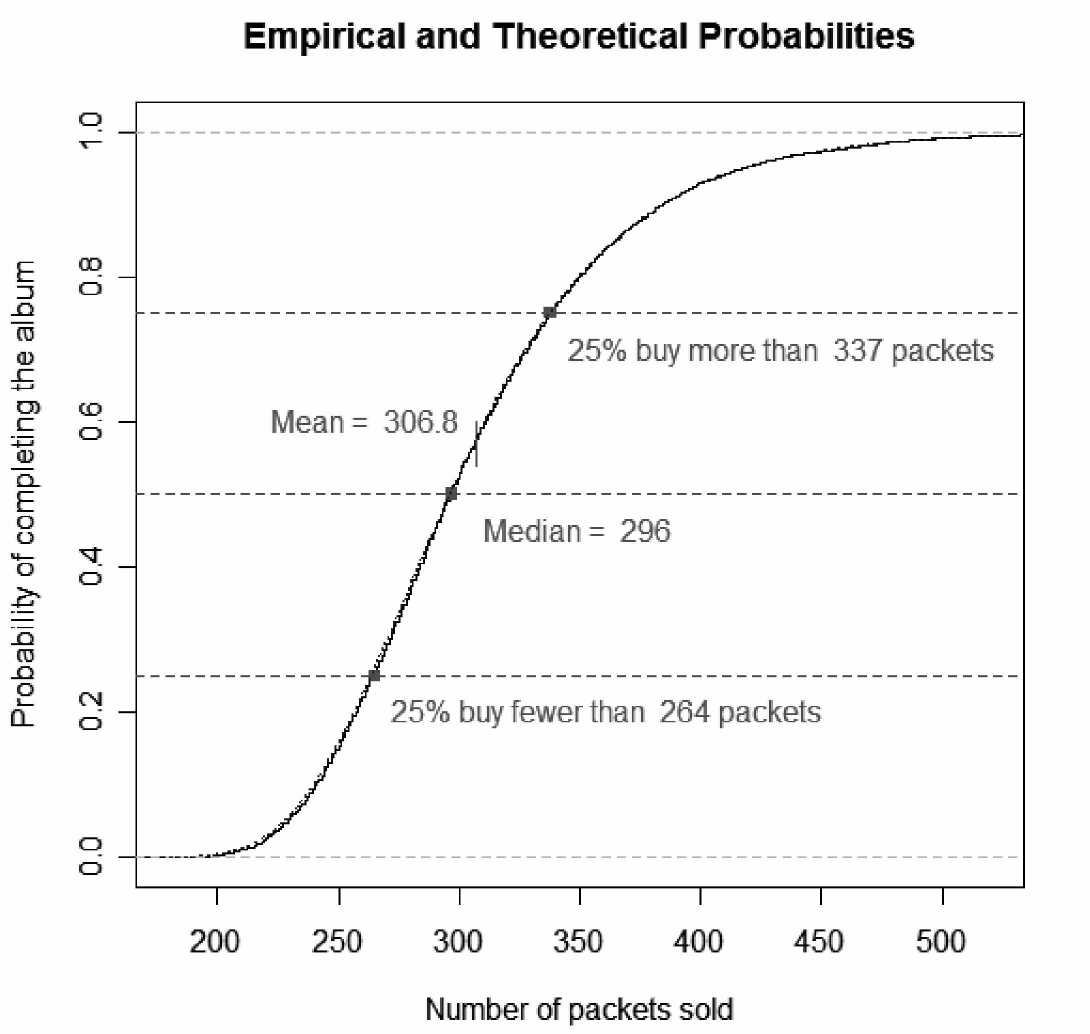


Figure 1. Empirical probability of 10,000 simulated GAA sticker albums, simulation summary statistics, and theoretical distribution function.

An article in Ireland on Sunday (June 11, 2006) used Figure 1 to report on the cost of filling the GAA stickers album. The headline of the article was based on the cost calculated using the upper quartile (give headline). The article warned that the costs arrived at excluded the possibility of sticker trading and indeed modelling the possibility of a trading arrangement with just one other collector can produce important downward adjustments in the estimated costs.

WEB Links

- Official GAA 2006 Players Stickers Album (see www.gaastickers.ie).
- Leonhard Euler (1707-1783, see http://www.amt.canberra.edu.au/euler.html for a brief online biography).

References

Dawkins, B. (1991). Siobhan's Problem: The Coupon Collector's Problem Revisited, *The American Statistician,* 45, 76-82.

R Development Core Team (2003). R: A Language and Environment for Statistical Computing, *Vienna, Austria: R Foundation for Statistical Computing.* www.r-project.org

Dr. Kevin Hayes and Ailish Hannigan
Department of Mathematics and Statistics,
University of Limerick, Limerick, Republic of Ireland.
EMAIL: kevin.hayes@ul.ie
TELEPHONE: 00353-(0)61-332388
FAX: 00353-(0)61-334927

A RECORD PRIME

The largest currently known prime number is the Mersenne prime $2^{32582657} - 1$, which was discovered in September 2006 and has 9808358 digits.
http://mathworld.wolfram.com/MersennePrime.html

GIMPS

The Great Internet Mersenne Prime Search

Finding record-breaking prime numbers is now a world-wide co-operative enterprise. Anyone can join in the fun by linking to www.mersenne.org/prime.htm. Finding large prime numbers can be rewarding in many ways—there is even a special prize awarded by the Electronic Frontier Foundation (http://www.eff.org/awards/coop.php)for the first ten million digit (at least) prime number. Happy hunting!

Leaving Certificate Mathematics Higher Level 2006—Paper 1

1. (a) Find the real number a such that for all $x \neq 9$,

$$\frac{x-9}{\sqrt{x}-3} = \sqrt{x} + a.$$

$$\frac{x-9}{\sqrt{x}-3} = \frac{x-9}{\sqrt{x}-3} \cdot \frac{\sqrt{x}+3}{\sqrt{x}+3}$$

$$= \frac{(x-9)(\sqrt{x}+3)}{(x-9)} = \sqrt{x}+3 \Rightarrow a = 3$$

OR

Let $\dfrac{x-9}{\sqrt{x}-3} = \sqrt{x}+a$

$$\Rightarrow \frac{x-9}{\sqrt{x}-3} = \frac{\sqrt{x}+a}{1}$$

$$\Rightarrow x-9 = (\sqrt{x}-3)(\sqrt{x}+a)$$

$$\Rightarrow x-9 = x - 3\sqrt{x} + a\sqrt{x} - 3a$$

$$\Rightarrow x + (0)(\sqrt{x}) + (-9) = x + (a-3)\sqrt{x} + (-3a)$$

Equating the coefficients we get

(i) $0 = a - 3$ or (ii) $-9 = -3a$

$\Rightarrow a = 3$ $\Rightarrow \underline{\underline{a = 3}}$

OR

$$\frac{x-9}{\sqrt{x}-3} = \frac{(\sqrt{x})^2 - (3)^2}{\sqrt{x}-3} = \frac{(\sqrt{x}-3)(\sqrt{x}+3)}{(\sqrt{x}-3)}$$

$$= \sqrt{x}+3$$

$$\Rightarrow \underline{\underline{a = 3}}$$

(b) $f(x) = 3x^3 + mx^2 - 17x + n$, where m and n are constants.
Given that $x - 3$ and $x + 2$ are factors of $f(x)$, find the value of m and the value of n.

Since $(x-3)$ is a factor of $f(x) \Rightarrow f(3) = 0$

$$f(3) = 3(3)^3 + m(3)^2 - 17(3) + n = 0$$

$$\Rightarrow 9m + n = -30 \ \ldots \ ①$$

Since $(x+2)$ is a factor $\Rightarrow f(-2)=0$

$f(-2)=3(-2)^3+m(-2)^2-17(-2)+n=0$

$\Rightarrow 4m+n=-10 \ldots$ ②

Solving equations ① and ②:

$$\begin{aligned} 9m+n&=-30\\ \underline{4m+n}&\underline{=-10}\\ 5m&=-20\\ \Rightarrow m&=-4 \end{aligned}$$

From (1): $9(-4)+n=-30$

$\Rightarrow n=6$

$\therefore\ \underline{\underline{m=-4 \text{ and } n=6}}$

OR

Since $(x+2)$ and $(x-3)$ are factors, then $(x+2)(x-3)$, i.e., (x^2-x-6) is a factor. The other factor must be linear i.e., $(3x+b)$.

$f(x)=3x^3+mx^2-17x+n=(x^2-x-6)(3x+b)$

$\Rightarrow 3x^3+(m)x^2+(-17)x+n=3x^3-3x^2-18x+bx^2-bx-6b$

$=3x^2+(b-3)x^2+(-b-18)x+(-6b)$

Equating the coefficients of both equations we get

(i) $m=b-3$

(ii) $-17=-b-18$

(iii) $n=-6b$

(ii) $-17=-b-18 \Rightarrow b=-1$

(i) $m=b-3 \Rightarrow m=-1-3 \Rightarrow m=-4$

(iii) $n=-6b \Rightarrow n=-6(-1) \Rightarrow n=6$

$\therefore\ \underline{\underline{m=-4 \text{ and } n=6}}$

(c) x^2-t is a factor of x^3-px^2-qx+r.

(i) Show that $pq=r$.

(ii) Express the roots of $x^3-px^2-qx+r=0$ in terms of p and q.

(i) If (x^2-t) is divided into x^3-px^2-qx+r, the remainder is zero.

$$\begin{array}{rl} & x-p \\ x^2-t & \overline{)\,x^3-px^2-qx+r} \\ & \underline{x^3 \qquad\quad -tx} \\ & \quad -px^2+(t-q)x+r \\ & \quad \underline{-px^2 \qquad\qquad\quad +pt} \end{array}$$

Remainder $=\quad (t-q)x+(r-pt)$

Since (x^2-t) is a factor, the remainder is zero

$\Rightarrow (t-q)x+(r-pt)=0(x)+0$

Equating coefficients, we have

(i) $t - q = 0$ and (ii) $r - pt = 0$

$\Rightarrow t = q$

From (ii) $r - pt = 0$

$\Rightarrow r = pt = pq$

i.e. $\underline{\underline{pq = r}}$

(ii) $f(x) = (x^2 - t)(x - p) = 0$

$\Rightarrow x^2 - t = 0$ or $x - p = 0$

$\Rightarrow x^2 = t$ or $x = p$

$\Rightarrow x = \pm\sqrt{t}$

$\Rightarrow x = \pm\sqrt{q}$

$\therefore$ the roots are $\underline{\underline{x = \pm\sqrt{q} \text{ or } x = p}}$

OR

(i) $f(x) = x^3 - px^2 - qx + r$

Since $(x^2 - t)$ is a factor, the other factor must be linear $= \left(x - \frac{r}{t}\right)$.

Thus $f(x) = x^3 + (-p)x^2 + (-q)x + r = (x^2 - t)\left(x - \frac{r}{t}\right) = x^3 - \frac{r}{t}x^2 - tx + r$

$$\Rightarrow x^3 + (-p)x^2 + (-q)x + r$$

$$= x^3 + \left(\frac{-r}{t}\right)x^2 + (-t)x + r$$

Equating coefficients, we have

(i) $-p = \frac{-r}{t}$

(ii) $-q = -t \Rightarrow t = q$

From (i) $-p = \frac{-r}{t} \Rightarrow r = pt \Rightarrow \underline{\underline{r = pq}}$

(ii) $f(x) = (x^2 - t)\left(x - \frac{r}{t}\right) = 0$

$\Rightarrow x^2 - t = 0$ or $x - \frac{r}{t} = 0$

$\Rightarrow x^2 = t$ or $x = \frac{r}{t}$

$x = \pm\sqrt{t}$ $\Rightarrow x = p$ ... from (i) above

$\Rightarrow x = \pm\sqrt{q}$

$\therefore$ $\underline{\underline{x = \pm\sqrt{q} \text{ or } x = p}}$

2. (a) Solve the simultaneous equations

$$y = 2x - 5.$$
$$x^2 + xy = 2.$$

Substitute $(2x - 5)$ for y in the equation $x^2 + xy = 2$

$x^2 + xy = 2 \Rightarrow x^2 + x(2x - 5) = 2$

$\Rightarrow x^2 + 2x^2 - 5x - 2 = 0$

$\Rightarrow 3x^2 - 5x - 2 = 0$

$\Rightarrow (3x + 1)(x - 2) = 0$

$\Rightarrow 3x + 1 = 0 \text{ or } x - 2 = 0$

$\Rightarrow x = -\frac{1}{3} \text{ or } x = 2$

$y = 2x - 5$

$x = -\frac{1}{3} : y = 2\left(-\frac{1}{3}\right) - 5 = -\frac{2}{3} - 5 = -5\frac{2}{3} = -\frac{17}{3}$

$x = 2 : y = 2(2) - 5 = -1$

The solutions are $x = -\frac{1}{3},\ y = -\frac{17}{3}$ or $x = 2,\ y = -1$

(b) (i) Find the range of values of $t \in \mathbf{R}$ for which the quadratic equation $(2t - 1)x^2 + 5tx + 2t = 0$ has real roots.

(ii) Explain why the roots are real when t is an integer.

(i) A quadratic equation has real roots when $b^2 - 4ac \geq 0$

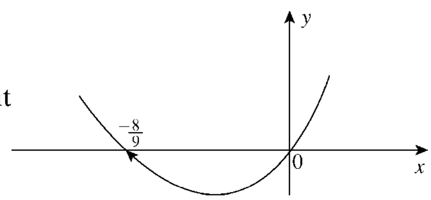
$(2t - 1)x^2 + 5tx + 2t = 0$

$b^2 - 4ac = (5t)^2 - 4(2t - 1)(2t)$

$= 25t^2 - 16t^2 + 8t \geq 0$

$\Rightarrow 9t^2 + 8t \geq 0$

Graph of $f(t) = 9t^2 + 8t$ is shown on the right



$9t^2 + 8t = 0$

$\Rightarrow t(9t + 8) = 0$

$\Rightarrow t = 0 \text{ or } t = -\frac{8}{9}$

The graph cuts the x-axis at $t = 0$ and $t = -\frac{8}{9}$

$\Rightarrow 9t^2 + 8t \geq 0$ when $t \leq -\frac{8}{9}$ and $t \geq 0$

$\therefore$ the roots are real for $t \leq -\frac{8}{9}$ and $t \geq 0$

(ii) Imaginary roots only when $-\frac{8}{9} < t < 0$.

Since there are no integers in this interval,
$\Rightarrow$the roots are real for all integers.

(c) $f(x) = 1 - b^{2x}$ and $g(x) = b^{1+2x}$, where b is a positive real number.

Find, in terms of b, the value of x for which $f(x) = g(x)$.

$f(x) = 1 - b^{2x} \qquad g(x) = b^{1+2x}$

$f(x) = g(x)$

$\Rightarrow 1 - b^{2x} = b^{1+2x}$

$\Rightarrow 1 - b^{2x} = b^1 \cdot b^{2x}$

$\Rightarrow \qquad 1 = b^{2x} + b^1 \cdot b^{2x}$

$\Rightarrow \qquad 1 = b^{2x}(1 + b)$

$\Rightarrow \frac{1}{1+b} = b^{2x}$

We now take the log of both sides to the base of b

$\Rightarrow \log_b\left(\frac{1}{b+1}\right) = \log_b(b^{2x})$

$\Rightarrow \log_b 1 - \log_b (b+1) = 2x\log_b b$

$\Rightarrow 0 - \log_b (b+1) = 2x(1)$

$\Rightarrow 2x = -\log_b (b+1)$

$\Rightarrow x = -\frac{1}{2}\log_b(b+1)$

$\Rightarrow x = -\log_b \sqrt{b+1}$

3. (a) Given that $z = 2 + i$, where $i^2 = -1$, find the real number d such that $z + \frac{d}{z}$ is real.

When $z + \frac{d}{z}$ is written as a real number,

$$z + \frac{d}{z} = (a) + (0)i, \ a \in \mathbf{R}$$

$$\frac{d}{z} = \frac{d}{2+i} = d\left[\frac{1}{2+i}\right]$$

$$= d\left[\frac{1}{2+i} \cdot \frac{2-i}{2-i}\right]$$

$$= d\left[\frac{2-i}{2^2-(i)^2}\right] = \frac{d}{5}(2-i)$$

$$z + \frac{d}{z} = 2 + i + \frac{d}{5}(2-i)$$

$$= 2 + i + \frac{2d}{5} - \frac{di}{5}$$

$$= \left(2 + \frac{2d}{5}\right) + \left(1 - \frac{d}{5}\right)i$$

$$z + \frac{d}{z} = a + 0i$$

$$\Rightarrow \left(2 + \frac{2d}{5}\right) + \left(1 - \frac{d}{5}\right)i = (a) + (0)i$$

Equating coefficients, we have

$$1 - \frac{d}{5} = 0 \Rightarrow 1 = \frac{d}{5} \Rightarrow \underline{\underline{d = 5}}$$

(b) (i) Use matrix methods to solve the simultaneous equations

$$4x - 2y = 5$$
$$8x + 3y = -4$$

(ii) Find the two values of k which satisfy the matrix equation

$$\begin{pmatrix} 1 & k \end{pmatrix} \begin{pmatrix} 3 & 4 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ k \end{pmatrix} = 11.$$

(i) $4x - 2y = 5$

$8x + 3y = -4$ $\Rightarrow \begin{pmatrix} 4 & -2 \\ 8 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ -4 \end{pmatrix}$

Let $A = \begin{pmatrix} 4 & -2 \\ 8 & 3 \end{pmatrix}$

$$\Rightarrow A^{-1} = \frac{1}{12 - (-16)} \begin{pmatrix} 3 & 2 \\ -8 & 4 \end{pmatrix} = \frac{1}{28} \begin{pmatrix} 3 & 2 \\ -8 & 4 \end{pmatrix}$$

$$A\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}5\\-4\end{pmatrix}$$

$$A^{-1}A\begin{pmatrix}x\\y\end{pmatrix} = A^{-1}\begin{pmatrix}5\\-4\end{pmatrix}$$

$$\Rightarrow \ \mathrm{I}\begin{pmatrix}x\\y\end{pmatrix} = \frac{1}{28}\begin{pmatrix}3 & 2\\-8 & 4\end{pmatrix}\begin{pmatrix}5\\-4\end{pmatrix}$$

$$\begin{pmatrix}x\\y\end{pmatrix} = \frac{1}{28}\begin{pmatrix}7\\-56\end{pmatrix}$$

$$= \begin{pmatrix}\frac{7}{28}\\ \frac{-56}{28}\end{pmatrix} = \begin{pmatrix}\frac{1}{4}\\-2\end{pmatrix}$$

$$\therefore \quad \underline{\underline{x = \frac{1}{4} \quad \text{and} \quad y = -2}}$$

(ii) $$(1 \quad k)\begin{pmatrix}3 & 4\\-2 & 1\end{pmatrix}\begin{pmatrix}1\\k\end{pmatrix} = 11$$

$$(1 \quad k)\begin{pmatrix}3+4k\\-2+k\end{pmatrix} = 11$$

$$\Rightarrow (3 + 4k) + (-2k + k^2) - 11 = 0$$

$$3 + 4k - 2k + k^2 - 11 = 0$$

$$\Rightarrow k^2 + 2k - 8 = 0$$

$$\Rightarrow (k+4)(k-2) = 0$$

$$\Rightarrow k+4 = 0 \ \text{ or } \ k-2 = 0$$

$$\Rightarrow \underline{\underline{k = -4 \ \text{ or } \ k = 2}}$$

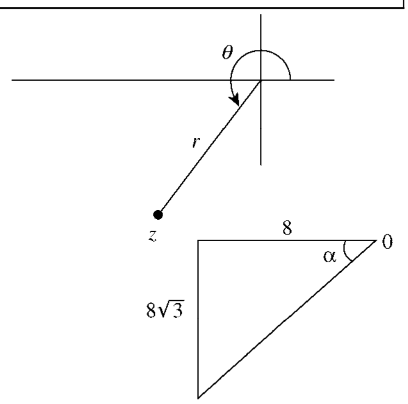
(c) (i) Express $-8-8\sqrt{3}i$ in the form $r(\cos\theta + i\sin\theta)$.

(ii) Hence find $(-8-8\sqrt{3}i)^3$.

(iii) Find the four complex numbers z such that

$$z^4 = -8-8\sqrt{3}i$$

Give your answers in the form $a+bi$, with a and b fully evaluated.



(i) $z = -8-8\sqrt{3}\,i$

$|z| = r$

$$r = \sqrt{(-8)^2 + (-8\sqrt{3})^2}$$

$$= \sqrt{64+192}$$

$$\Rightarrow r = \sqrt{256} \Rightarrow r = 16$$

$$z = r(\cos\theta + i\sin\theta)$$

$$= 16\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$$

$$= 2^4\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$$

$$\tan\alpha = \frac{8\sqrt{3}}{8} = \sqrt{3}$$

$$\Rightarrow \alpha = 60^\circ = \frac{\pi}{3}$$

$$\theta = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$$

(ii) $z = 2^4\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$

$$z^3 = \left[2^4\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)\right]^3$$

$$= 2^{12}(\cos 4\pi + i\sin 4\pi)$$

$$= 2^{12}(1 + i(0))$$

$$= 2^{12}$$

$$= \underline{\underline{4096}}$$

(iii) $z^4 = -8 - 8\sqrt{3}i = 2^4\left[\cos\left(2n\pi + \frac{4\pi}{3}\right) + i\sin\left(2n\pi + \frac{4\pi}{3}\right)\right]$

$$\Rightarrow z = \left\{2^4\left[\cos\left(2n\pi + \frac{4\pi}{3}\right) + i\sin\left(2n\pi + \frac{4\pi}{3}\right)\right]\right\}^{\frac{1}{4}}$$

$$\Rightarrow z = 2\left[\cos\left(\frac{n\pi}{2} + \frac{\pi}{3}\right) + i\sin\left(\frac{n\pi}{2} + \frac{\pi}{3}\right)\right]$$

Let $n = 0$: $z_0 = 2\left[\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right]$

$$= 2\left[\frac{1}{2} + i\left(\frac{\sqrt{3}}{2}\right)\right]$$

$$\Rightarrow z_0 = 1 + i\sqrt{3}$$

$n = 1$: $z_1 = 2\left[\cos\left(\frac{\pi}{2} + \frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{2} + \frac{\pi}{3}\right)\right]$

$$= 2\left[\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right]$$

$$= 2\left[-\frac{\sqrt{3}}{2} + i\left(\frac{1}{2}\right)\right]$$

$$\Rightarrow z_1 = -\sqrt{3} + i$$

$n = 2$: $z_2 = 2\left[\cos\left(\pi + \frac{\pi}{3}\right) + i\sin\left(\pi + \frac{\pi}{3}\right)\right]$

$$= 2\left[\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right]$$

$$= 2\left[-\frac{1}{2} - i\left(\frac{\sqrt{3}}{2}\right)\right]$$

$$\Rightarrow z_2 = -1 - i\sqrt{3}$$

$n = 3$: $z_3 = 2\left[\cos\left(\frac{3\pi}{2} + \frac{\pi}{3}\right) + i\sin\left(\frac{3\pi}{2} + \frac{\pi}{3}\right)\right]$

$$= 2\left[\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right]$$

$$= 2\left[\frac{\sqrt{3}}{2} + i\left(-\frac{1}{2}\right)\right]$$

$$\Rightarrow z_3 = \sqrt{3} - i$$

$\therefore$ the 4 roots are: $\underline{\underline{1 + i\sqrt{3}, -\sqrt{3} + i, -1 - i\sqrt{3}, \sqrt{3} - i}}$

4. (a) $-2+2+6+\cdots+(4n-6)$ are the first n terms of an arithmetic series.
S_n, the sum of these n terms, is 160. Find the value of n.

$-2+2+6+\cdots+(4n-6)$

$\Rightarrow a=-2$ and $d=4$; $S_n=160\cdots$ (given)

$$S_n=\frac{n}{2}[2a+(n-1)d]=160$$

$$\Rightarrow\frac{n}{2}[2(-2)+(n-1)4]=160$$

$$\Rightarrow\frac{n}{2}[-4+4n-4]=160$$

$$\Rightarrow n(2n-4)=160$$

$$\Rightarrow 2n^2-4n-160=0$$

$$\Rightarrow n^2-2n-80=0$$

$$\Rightarrow (n-10)(n+8)=0$$

$$\Rightarrow n-10=0 \quad \text{or} \quad n+8=0$$

$$\Rightarrow n=10 \quad \text{or} \quad n=-8$$

$n\neq-8\Rightarrow \underline{\underline{n=10}}$

(b) The sum to infinity of a geometric series is $\frac{9}{2}$.
The second term of the series is –2.
Find the value of r, the common ratio of the series.

$$S_\infty=\frac{a}{1-r}=\frac{9}{2}$$

$$\Rightarrow 9(1-r)=2a\cdots \quad ①$$

Series: $a,\ ar,\ ar^2$

$$\Rightarrow ar=-2 \quad\Rightarrow\quad a=\frac{-2}{r}\cdots \quad ②$$

①: $9(1-r) = 2a$

$\Rightarrow 9(1-r) = 2\left(\frac{-2}{r}\right)$

$\Rightarrow 9 - 9r = \left(\frac{-4}{r}\right)$

$\Rightarrow 9r - 9r^2 = -4$

$\Rightarrow 9r^2 - 9r - 4 = 0$

$\Rightarrow (3r+1)(3r-4) = 0$

$\Rightarrow 3r + 1 = 0 \quad \text{or} \quad 3r - 4 = 0$

$\Rightarrow r = -\frac{1}{3} \quad \text{or} \quad r = \frac{4}{3}$

$\Rightarrow \underline{\underline{r = -\frac{1}{3}}} \ldots$ [as $|r| < 1$].

(c) The sequence $u_1, u_2, u_3, \ldots,$ defined by $u_1 = 3$ and $u_{n+1} = 2u_n + 3$ is as follows: 3, 9, 21, 45, 93 ...

(i) Find u_6, and verify that it is equal to the sum of the first six terms of a geometric series with first term 3 and common ratio 2.

(ii) Given that, for all k, u_k is the sum of the first k terms of a geometric series with first term 3 and common ratio 2, find $\sum_{k=1}^{n} u_k$.

(i) $u_1 = 3 \qquad u_{n+1} = 2u_n + 3$

$u_6 = 2u_5 + 3$

$= 2(93) + 3$

$= 189$

Geometric sequence: $a = 3$ and $r = 2$

$S_n = \frac{a(r^n - 1)}{r - 1} \ldots |r| > 1$

$S_6 = \frac{3(2^6 - 1)}{2 - 1} = 3(64 - 1) = 189$

$\Rightarrow \underline{\underline{u_6 = S_6 = 189}}$

(ii) The new series is

$$3,\ (3+6),(3+6+12),(3+6+12+24),+\cdots$$

i.e. $3, 9, 21, 45, 93, \ldots$

$u_k = (3+6+12+\cdots \text{ to } k \text{ terms})$

Here $a=3,\ r=2;\ n=k$

$$\Rightarrow u_k = \frac{a(r^n-1)}{r-1} = \frac{3[2^k-1]}{2-1} = 3[2^k-1] = 3(2^k)-3$$

$$u_n = 3(2^n)-3$$
$$u_{n-1} = 3(2^{n-1})-3$$
$$u_{n-2} = 3(2^{n-2})-3$$
$$\cdots\cdots\ \cdots\cdots\cdots$$
$$\cdots\cdots\ \cdots\cdots\cdots$$
$$u_3 = 3(2^3)-3$$
$$u_2 = 3(2^2)-3$$
$$u_1 = 3(2^1)-3$$

$$\sum_{k=1}^{n} u_k = 3[2^1+2^2+2^3+\cdots+2^n]-3(n)$$

$= 3[\text{Geometric series with } a=2,\ r=2 \text{ and } n=n]-3n$

$$= 3\left[\frac{2(2^n-1)}{2-1}\right]-3n$$

$$= 3[2(2^n-1)]-3n$$

$$= \underline{\underline{6(2^n-1)-3n}}$$

5. (a) Find the value of the middle term of the binomial expansion of

$$\left(\frac{x}{y}-\frac{y}{x}\right)^8.$$

$$\left(\frac{x}{y}-\frac{y}{x}\right)^8 = \left[\frac{x}{y}+\left(\frac{-y}{x}\right)\right]^8$$

There are 9 terms in the expansion $\Rightarrow T_5$ is the middle term.

$$u_5 = \binom{8}{4}\left(\frac{x}{y}\right)^4\left(-\frac{y}{x}\right)^4$$

$$= \frac{8.7.6.5}{4.3.2.1}\left[\frac{x^4}{y^4}\cdot\frac{y^4}{x^4}\right]$$

$$= \underline{\underline{70}}$$

OR

$$\left[\frac{x}{y}+\left(\frac{-y}{x}\right)\right]^8=\left(\frac{x}{y}\right)^8+\binom{8}{1}\left(\frac{x}{y}\right)^7\left(\frac{-y}{x}\right)+\binom{8}{2}\left(\frac{x}{y}\right)^6\left(\frac{-y}{x}\right)^2$$

$$+\binom{8}{3}\left(\frac{x}{y}\right)^5\left(\frac{-y}{x}\right)^3+\binom{8}{4}\left(\frac{x}{y}\right)^4\left(\frac{-y}{x}\right)^4$$

$$\Rightarrow u_5=\binom{8}{4}\left(\frac{x}{y}\right)^4\left(\frac{-y}{x}\right)^4$$

$$=\frac{8.7.6.5}{4.3.2.1}\cdot\frac{x^4}{y^4}\cdot\frac{y^4}{x^4}$$

$$=\underline{\underline{70}}$$

(b) (i) Express $\frac{2}{(r+1)(r+3)}$ in the form $\frac{A}{r+1}+\frac{B}{r+3}$.

(ii) Hence find $\sum_{r=1}^{n}\frac{2}{(r+1)(r+3)}$.

(iii) Hence evaluate $\sum_{r=1}^{\infty}\frac{2}{(r+1)(r+3)}$.

(i) $\frac{A}{r+1}+\frac{B}{r+3}=\frac{2}{(r+1)(r+3)}$

Multiply across by $(r+1)(r+3)$

$$\Rightarrow A(r+3)+B(r+1)=2$$

$$\Rightarrow Ar+3A+Br+B=2$$

$$\Rightarrow Ar+Br+3A+B=2$$

$$\Rightarrow (A+B)r+(3A+B)=0(r)+2$$

Equating coefficients we have

$$A+B=0\Rightarrow A=-B$$

and $3A+B=2$

$$\Rightarrow 3A-A=2$$

$$\Rightarrow 2A=2 \Rightarrow A=1 \Rightarrow B=-1$$

$$\therefore \frac{2}{(r+1)(r+3)}=\underline{\underline{\frac{1}{r+1}-\frac{1}{r+3}}}$$

(ii) To find $\sum_{r=1}^{n} \frac{2}{(r+1)(r+3)}$

$$u_n = \frac{1}{n+1} - \frac{1}{n+3}$$

$$u_{n-1} = \frac{1}{n} - \frac{1}{n+2}$$

$$u_{n-2} = \frac{1}{(n-1)(n+1)} - \frac{1}{n+1}$$

…… ………

…… ………

$$u_3 = \frac{1}{4} - \frac{1}{6}$$

$$u_2 = \frac{1}{3} - \frac{1}{5}$$

$$u_1 = \frac{1}{2} - \frac{1}{4}$$

$$S_n = \frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3}$$

$$= \frac{5}{6} - \frac{1}{n+2} - \frac{1}{n+3}$$

(iii) As $n \to \infty$, $\lim S_n = \frac{5}{6} - 0 - 0$

$$= \frac{5}{6}$$

> (c) (i) Given two real numbers a and b, where $a > 1$ and $b > 1$, prove that
>
> $$\frac{1}{\log_b a} + \frac{1}{\log_a b} \geq 2.$$
>
> (ii) Under what condition is $\frac{1}{\log_b a} + \frac{1}{\log_a b} = 2$.

(i) To prove $\frac{1}{\log_b a} + \frac{1}{\log_a b} \geq 2$

$$\Rightarrow \log_a b + \frac{1}{\log_a b} \geq 2 \cdots \left[\log_a b = \frac{1}{\log_b a}\right]$$

Multiply across by $\log_a b > 0$.

$$(\log_a b)^2 + 1 \geq 2\log_a b$$

$$(\log_a b)^2 - 2\log_a b + 1 \geq 0$$

$$= (\log_a b - 1)^2 \geq 0 \ \cdots \ \text{true}$$

$$\Rightarrow \frac{1}{\log_b a} + \frac{1}{\log_a b} \geq 2$$

(ii) When $a = b \cdots (a > 0, \ b > 0)$

$$\frac{1}{\log_a b} + \frac{1}{\log_b a} = \frac{1}{\log_b b} + \frac{1}{\log_a a} = \frac{1}{1} + \frac{1}{1} = 2$$

Thus $\frac{1}{\log_b a} + \frac{1}{\log_a b} = 2$ when $a = b$

OR

In (i) above we found that

$$\frac{1}{\log_b a} + \frac{1}{\log_a b} \geq 2 \text{ is equivalent to } (\log_a b - 1)^2 \geq 0$$

Now let $(\log_a b - 1)^2 = 0$

$$\Rightarrow \log_a b - 1 = 0$$

$$\Rightarrow \log_a b = 1$$

This can only happen when $b = a$.

6. (a) Differentiate $\sqrt{x}(x+2)$ with respect to x.

Let $y = \sqrt{x}(x+2)$

$$= x^{\frac{1}{2}}(x+2)$$

$$\Rightarrow y = x^{\frac{3}{2}} + 2x^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{3}{2}(x^{\frac{1}{2}}) + 2.\frac{1}{2}(x^{-\frac{1}{2}})$$

$$= \frac{3}{2}\sqrt{x} + \frac{1}{\sqrt{x}}$$

OR

$$y = x^{\frac{1}{2}}(x+2) \cdots = uv$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$= (x^{\frac{1}{2}})(1) + (x+2)\left(\frac{1}{2}x^{-\frac{1}{2}}\right)$$

$$= x^{\frac{1}{2}} + \frac{x+2}{2x^{\frac{1}{2}}}$$

$$= \sqrt{x} + \frac{x+2}{2\sqrt{x}}$$

(b) The equation of a curve is $y = 3x^4 - 2x^3 - 9x^2 + 8$.

(i) Show that the curve has a local maximum at the point (0, 8).

(ii) Find the coordinates of the two local minimum points on the curve.

(iii) Draw a sketch of the curve.

(i) $y = 3x^4 - 2x^3 - 9x^2 + 8$

$$\frac{dy}{dx} = 12x^3 - 6x^2 - 18x$$

$$\frac{d^2y}{dx^2} = 36x^2 - 12x - 18$$

Local max/min: $\frac{dy}{dx} = 0 \Rightarrow 12x^3 - 6x^2 - 18x = 0$

$$\Rightarrow 2x^3 - x^2 - 3x = 0$$

$$\Rightarrow x(2x^2 - x - 3) = 0$$

$$\Rightarrow x(2x-3)(x+1) = 0$$

$$\Rightarrow x = 0, \quad x = \frac{3}{2} \quad \text{or} \quad x = -1$$

There are turning points at $x = 0$, $x = \frac{3}{2}$ and $x = -1$.

We now use the $\frac{d^2y}{dx^2}$ test for maximum or minimum values.

$$\frac{d^2y}{dx^2} = 36x^2 - 12x - 18$$

At $x = 0$, $\frac{d^2y}{dx^2} = 0 - 0 - 18 = -18 < 0$

$\therefore$ maximum turning point at $x = 0$

At $x = 0$, $y = f(x) = 3x^4 - 2x^3 - 9x^2 + 8$

$$= 0 - 0 - 0 + 8 \quad \text{at} \quad x = 0$$

$$= 8$$

Therefore local maximum at (0, 8).

(ii) We now test for local minima at $x = \frac{3}{2}$ and $x = -1$.

$$x = \frac{3}{2}: \quad y = 3x^4 - 2x^3 - 9x^2 + 8$$

$$= 3\left(\frac{3}{2}\right)^4 - 2\left(\frac{3}{2}\right)^3 - 9\left(\frac{3}{2}\right)^2 + 8$$

$$= \frac{243}{16} - \frac{108}{16} - \frac{324}{16} + \frac{128}{16}$$

$$= \frac{-61}{16} = -3.8 \quad \Rightarrow \left(\frac{3}{2}, -3.8\right)$$

Test $\frac{d^2y}{dx^2} = 36x^2 - 12x - 18$

$$= 36\left(\frac{3}{2}\right)^2 - 12\left(\frac{3}{2}\right) - 18 \text{ at } x = \frac{3}{2}$$

$$= 81 - 18 - 18 > 0$$

$\therefore$ Local minimum at $\left(\frac{3}{2}, \frac{-61}{16}\right)$

$$x = -1 \Rightarrow y = 3x^4 - 2x^3 - 9x^2 + 8$$

$$= 3(-1)^4 - 2(-1)^3 - 9(-1)^2 + 8$$

$$= 3 + 2 - 9 + 8$$

$$= 4 \qquad \Rightarrow (-1, 4)$$

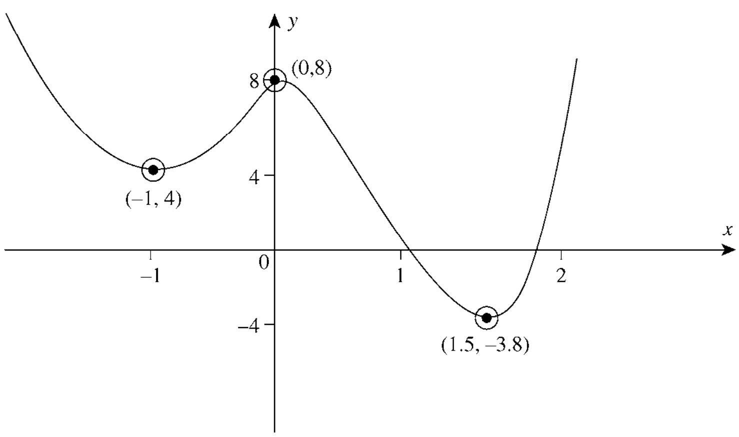
At $x = -1$, $\frac{d^2y}{dx^2} = 36x^2 - 12x - 18$

$$= 36(-1)^2 - 12(-1) - 18$$

$$= 36 + 12 - 18 > 0$$

$\therefore$ Local minimum at $(-1, 4)$.

(iii) Sketch of curve:



(c) Prove by induction that $\frac{d}{dx}(x^n) = nx^{n-1},\ n \geq 1,\ n \in \mathbf{N}$.

$$P(n):\quad \frac{d}{dx}(x^n) = nx^{n-1},\ n \geq 1,\ n \in \mathbf{N}$$

$$P(1):\quad \frac{d}{dx}(x^1) = 1.x^0 = 1$$

$$f(x) = x$$
$$f(x+h) = x+h$$
$$f(x+h) - f(x) = h$$
$$\frac{f(x+h) - f(x)}{h} = 1$$
$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = 1 \qquad \Rightarrow\ P(1) \text{ is true}$$

Assume $P(k)$ is true: $\frac{d}{dx}(x^k) = kx^{k-1}$

Test $P(k+1)$: $\frac{d}{dx}(x^{k+1}) = \frac{d}{dx}(x^k . x^1)$

$$= x^k(1) + x\ (kx^{k-1})$$
$$= x^k + kx^k$$
$$= (k+1)\ x^k$$

Therefore true for $P(k+1)$
As $P(1)$ is true, it is true for all $n \geq 1$.

7. (a) Taking $x_1 = 2$ as the first approximation to the real root of the equation

$$x^3 + x - 9 = 0,$$

use the Newton–Raphson method to find x_2, the second approximation.

Newton–Raphson formula: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$f(x) = x^3 + x - 9$: $f(x_1) = f(2) = 2^3 + 2 - 9 = 1$

$f'(x) = 3x^2 + 1$: $f'(x_1) = f'(2) = 3(2)^2 + 1 = 13$

$$x_2 = 2 - \frac{1}{13} = \underline{\underline{\frac{25}{13}}}$$

(b) The parametric equations of a curve are:

$x = 3\cos\theta - \cos^3\theta$

$y = 3\sin\theta - \sin^3\theta$, where $0 < \theta < \frac{\pi}{2}$.

(i) Find $\frac{dy}{d\theta}$ and $\frac{dx}{d\theta}$.

(ii) Hence show that $\frac{dy}{dx} = \frac{-1}{\tan^3\theta}$.

(i) $x = 3\cos\theta - \cos^3\theta$

$$\frac{dx}{d\theta} = -3\sin\theta - 3\cos^2\theta\,(-\sin\theta)$$

$$= -3\sin\theta + 3\sin\theta\cos^2\theta$$

$$= -3\sin\theta(1 - \cos^2\theta)$$

$$= -3\sin\theta\,(\sin^2\theta)$$

$$\Rightarrow \frac{dx}{d\theta} = \underline{\underline{-3\sin^3\theta}}$$

$y = 3\sin\theta - \sin^3\theta$

$$\frac{dy}{d\theta} = 3\cos\theta - 3\sin^2\theta\cos\theta$$

$$= 3\cos\theta\,(1 - \sin^2\theta)$$

$$= 3\cos\theta\,(\cos^2\theta)$$

$$\Rightarrow \frac{dy}{d\theta} = \underline{\underline{3\cos^3\theta}}$$

(ii) $$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3\cos^3\theta}{-3\sin^3\theta}$$

$$= \frac{-1}{\frac{\sin^3\theta}{\cos^3\theta}} = \underline{\underline{-\frac{1}{\tan^3\theta}}}$$

(c) Given $y = \ln\left(\frac{3 + x}{\sqrt{9 - x^2}}\right)$, find $\frac{dy}{dx}$ and express it in the form $\frac{a}{b - x^n}$.

$$y = \ln\left(\frac{3 + x}{\sqrt{9 - x^2}}\right)$$

$$= \ln(3 + x) - \ln(\sqrt{9 - x^2})$$

$$= \ln(3 + x) - \frac{1}{2}\ln(9 - x^2)$$

$$\frac{dy}{dx} = \frac{1}{3 + x}(1) - \frac{1}{2}\left[\frac{1}{9 - x^2}.(-2x)\right]$$

$$= \frac{1}{3+x} + \frac{x}{9-x^2}$$

$$= \frac{1}{3+x} + \frac{x}{(3+x)(3-x)}$$

$$= \frac{3-x+x}{(3+x)(3-x)} = \frac{3}{(3+x)(3-x)} = \underline{\underline{\frac{3}{9-x^2}}}$$

OR

$$y = \ln\left[\frac{3+x}{\sqrt{9-x^2}}\right]$$

$$= \ln\frac{3+x}{\sqrt{(3+x)(3-x)}} = \ln\left[\frac{(3+x)^{\frac{1}{2}}}{(3-x)^{\frac{1}{2}}}\right]$$

$$= \ln(3+x)^{\frac{1}{2}} - \ln(3-x)^{\frac{1}{2}}$$

$$\Rightarrow \quad y = \frac{1}{2}\ln(3+x) - \frac{1}{2}\ln(3-x)$$

$$\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{3+x} + \frac{1}{3-x}\right]$$

$$= \frac{1}{2}\left[\frac{3-x+3+x}{(3+x)(3-x)}\right]$$

$$= \frac{1}{2}\left[\frac{6}{9-x^2}\right] = \underline{\underline{\frac{3}{9-x^2}}}$$

8. (a) Find (i) $\int \sqrt{x}\,dx$ (ii) $\int e^{-2x}dx$.

(i) $\int \sqrt{x}\,dx = \int x^{\frac{1}{2}}dx = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c = \underline{\underline{\frac{2}{3}x^{\frac{3}{2}} + c}}$

(ii) $\int e^{-2x}dx = \underline{\underline{-\frac{1}{2}e^{-2x} + c}}$

(b) Evaluate (i) $\int_1^2 x(1+x^2)^3 dx$ (ii) $\int_0^{\frac{\pi}{4}} \sin 5\theta \cos 3\theta \, d\theta$

(i) $\int_1^2 x(1+x^2)^3 dx$

Let $u = 1 + x^2$
$du = 2x\,dx$
$\Rightarrow xdx = \frac{1}{2}du$

$$= \int u^3 . \frac{1}{2} du$$

$$= \frac{1}{2}\int u^3 du$$

$$= \frac{1}{2}\left[\frac{u^4}{4}\right]$$

$$= \frac{1}{2}\left[\frac{(1+x^2)^4}{4}\right]_1^2 = \frac{1}{8}\left[\left(1+x^2\right)^4\right]_1^2$$

$$= \frac{1}{8}[(5)^4 - (2)^4] = \frac{609}{8} = 76\frac{1}{8}$$

OR

$$(1+x^2)^3 = 1 + \binom{3}{1}x^2 + \binom{3}{2}(x^2)^2 + \binom{3}{3}(x^2)^3$$
$$= 1 + 3x^2 + 3x^4 + x^6$$
$$x(1+x^2)^3 = x + 3x^3 + 3x^5 + x^7$$

$$\int_1^2 (x + 3x^3 + 3x^5 + x^7)dx$$

$$= \left[\frac{x^2}{2} + \frac{3x^4}{4} + \frac{x^6}{2} + \frac{x^8}{8}\right]_1^2$$

$$= [2 + 12 + 32 + 32] - \left[\frac{1}{2} + \frac{3}{4} + \frac{1}{2} + \frac{1}{8}\right]$$

$$= 78 - 1\frac{7}{8} = 76\frac{1}{8}$$

(ii) $\int_0^{\frac{\pi}{4}} \sin 5\theta \cos 3\theta \, d\theta$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} 2\sin 5\theta \cos 3\theta d\theta$$

$$= \frac{1}{2}\int (\sin 8\theta + \sin 2\theta) d\theta$$

$$= \frac{1}{2}\left[-\frac{\cos 8\theta}{8} - \frac{\cos 2\theta}{2}\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2}\left[\left(\frac{-\cos 2\pi}{8} - \frac{\cos\frac{\pi}{2}}{2}\right) - \left(\frac{-\cos 0}{8} - \frac{\cos 0}{2}\right)\right]$$

$$= \frac{1}{2}\left[\left(-\frac{1}{8} - 0\right) - \left(-\frac{1}{8} - \frac{1}{2}\right)\right]$$

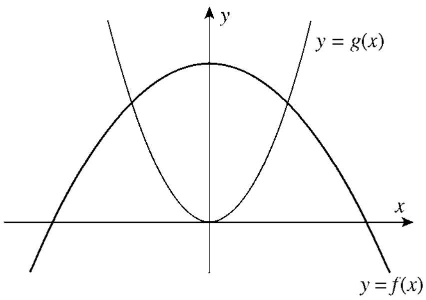
$$= \frac{1}{2}\left[-\frac{1}{8} + \frac{1}{8} + \frac{1}{2}\right]$$

$$= \frac{1}{4}$$

(c) The diagram shows the graphs of the curves $y = f(x)$ and $y = g(x)$, where $f(x) = 12 - 3x^2$ and $g(x) = 9x^2$.

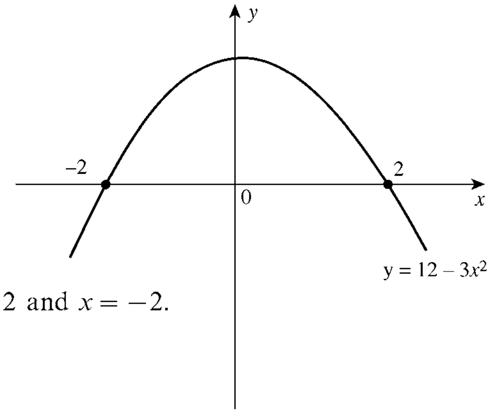
(i) Calculate the area of the region enclosed by the curve $y = f(x)$ and the x-axis.

(ii) Show that the region enclosed by the curves $y = f(x)$ and $y = g(x)$ has half that area.



(i) $f(x) = 12 - 3x^2$

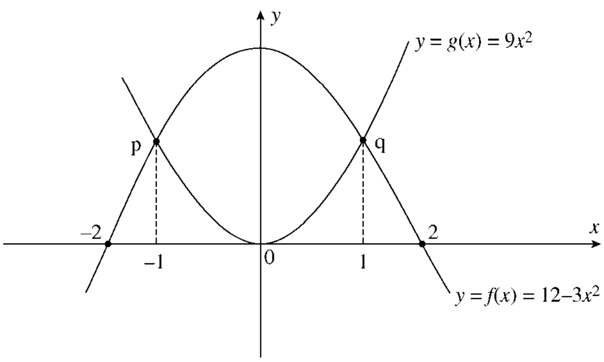
$$f(x) = 0 \Rightarrow 12 - 3x^2 = 0$$
$$\Rightarrow 4 - x^2 = 0$$
$$\Rightarrow x^2 = 4$$
$$\Rightarrow x = \pm 2$$



The curve intersects the x-axis at $x = 2$ and $x = -2$.

$$\text{Area, } A = \int_{-2}^{2} y\,dx = 2\int_{0}^{2} y\,dx$$
$$= 2\int (12 - 3x^2)dx$$
$$= 2[12x - x^3]_0^2$$
$$= 2[(24 - 8) - (0 - 0)]$$
$$= \underline{\underline{32}}$$

(ii)



The two curves intersect when $f(x) = g(x)$.

$$12 - 3x^2 = 9x^2$$
$$\Rightarrow 12 = 12x^2$$
$$\Rightarrow 1 = x^2 \Rightarrow x = \pm 1$$

$$\text{Enclosed area} = \int_{-1}^{1} f(x)dx \; - \; \int_{-1}^{1} g(x)dx$$

$$= 2\left[\int_0^1 (12 - 3x^2)dx \; - \; \int_0^1 9x^2 dx\right]$$

$$= 2[12x - x^3 - 3x^3]_0^1$$

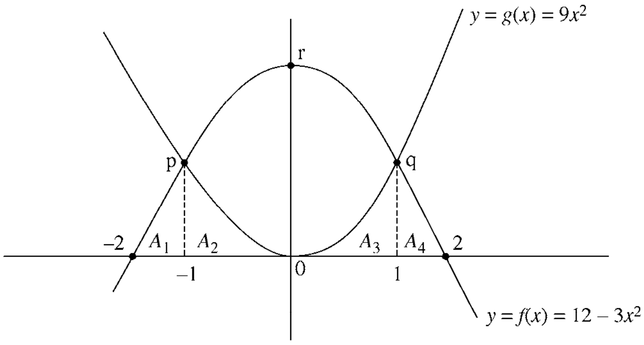
$$= 2[12x - 4x^3]_0^1$$

$$= 2[(12 - 4) - (0 - 0)]$$

$$= 2[8]$$

$$= 16 \cdots = \frac{1}{2} \text{ of } 32, \text{ found in part (i)}$$

OR



The curves intersect when $f(x) = g(x)$

$$12 - 3x^2 = 9x^2$$

$$\Rightarrow 12 = 12x^2$$

$$\Rightarrow \; 1 = x^2 \Rightarrow x = \pm 1$$

$$\text{Enclosed area} = 32 - [A_1 + A_2 + A_3 + A_4]$$

$$= 32 - 2(A_3 + A_4)$$

$$A_3 = \int_0^1 9x^2 dx \;=\; [3x^3]_0^1 = 3(1) - 0 = 3$$

$$A_4 = \int_1^2 (12 - 3x^2)dx \;=\; [12x - x^3]_1^2$$

$$= [24 - 8] - [12 - 1]$$

$$= 16 - 11$$

$$= 5$$

$$2(A_3 + A_4) = 2(3 + 5) = 16$$

$$\text{Enclosed area} = 32 - 2(A_3 + A_4)$$

$$= 32 - 16$$

$$= 16 \cdots = \frac{1}{2} \text{ of } 32$$

Leaving Certificate Mathematics
Higher Level 2006—Paper 2

1. (a) $a(-1,-3)$ and $b\ (3,1)$ are the end-points of a diameter of a circle.
Write down the equation of the circle.

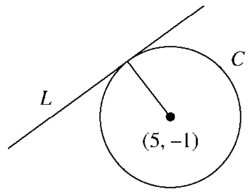
Mid-point of $[ab]$ = Centre of circle $c = (1, -1)$
Radius $= |ac| = \sqrt{4+4} = \sqrt{8}$.

$\therefore$ Equation of circle: $(x-1)^2 + (y+1)^2 = 8$.

(b) Circle C has centre $(5, -1)$.
The line L: $3x - 4y + 11 = 0$ is a tangent to C.

(i) Show that the radius of C is 6.

(ii) The line $x + py + 1 = 0$ is also a tangent to C.
Find two possible values of p.



(i) Radius = Distance from centre $(5, -1)$ to line $3x - 4y + 11 = 0$.

$$\text{Radius} = \left| \frac{15 + 4 + 11}{\sqrt{9 + 16}} \right| = \underline{\underline{6}}.$$

(ii) Perpendicular distance from centre $(5, -1)$ to $x + py + 1 = 0$ equals 6.

$$\therefore \left| \frac{5 - p + 1}{\sqrt{1 + p^2}} \right| = 6 \Rightarrow (6 - p)^2 = 36 + 36p^2.$$

$$\therefore 35p^2 + 12p = 0 \Rightarrow p(35p + 12) = 0$$

$$\Rightarrow p = 0 \text{ or } p = -\frac{12}{35}$$

(c) S is the circle $x^2 + y^2 + 4x + 4y - 17 = 0$ and K is the line $4x + 3y = 12$.

(i) Show that the line K does not intersect S.

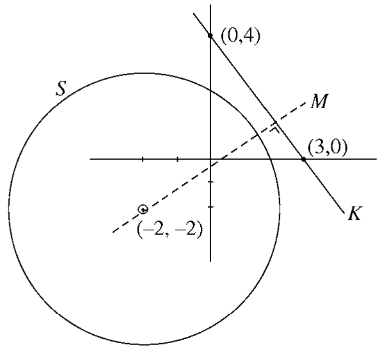
(ii) Find the co-ordinates of the point on S that is closest to K.

(i) Centre $= (-2, -2)$, radius $= \sqrt{g^2 + f^2 - c} = \sqrt{4 + 4 + 17} = 5$.

$$\text{Distance from line to centre} = \left| \frac{-8 - 6 - 12}{5} \right| = \frac{26}{5} > 5.$$

$\therefore$ K does not intersect the circle S.

(ii) Line $M \perp$ line K
Equation of M: $3x - 4y + k = 0$
$(-2, -2) \in M \Rightarrow -6 + 8 + k = 0$
$\Rightarrow \quad k = -2$
Equation of M: $3x - 4y - 2 = 0$



Required point is $3x - 4y = 2 \cap S$.

$$x = \frac{2+4y}{3} \Rightarrow \left(\frac{2+4y}{3}\right)^2 + y^2 + 4\left(\frac{2+4y}{3}\right) + 4y - 17 = 0.$$

$$\therefore \frac{4+16y+16y^2}{9} + y^2 + \frac{8+16y}{3} + 4y - 17 = 0.$$

$$4 + 16y + 16y^2 + 9y^2 + 24 + 48y + 36y - 153 = 0$$

$$\Rightarrow 25y^2 + 100y - 125 = 0.$$

$$\therefore y^2 + 4y - 5 = 0 \Rightarrow (y-1)(y+5) = 0 \Rightarrow y = 1 \text{ or } y = -5.$$

$$\therefore (2, 1) \text{ or } (-6, -5).$$

But (2,1) is closest point.
$\therefore$ Solution = $\underline{\underline{(2,1)}}$.

2. (a) $\vec{x} = -3\vec{i} + \vec{j}$. Express $(\vec{x}^{\perp})^{\perp}$ in terms of $\vec{i}$ and $\vec{j}$

$$\vec{x} = -3\vec{i} + \vec{j} \Rightarrow \vec{x}^{\perp} = (-3\vec{i} + \vec{j})^{\perp} = -\vec{i} - 3\vec{j} \Rightarrow (\vec{x}^{\perp})^{\perp} = \underline{\underline{3\vec{i} - \vec{j}}}.$$

(b) $\vec{p} = -5\vec{i} + 2\vec{j}$, $\vec{q} = \vec{i} - 6\vec{j}$ and $\vec{r} = -\vec{i} + 5\vec{j}$.
(i) Express $\overrightarrow{pq}$ and $\overrightarrow{pr}$ in terms of $\vec{i}$ and $\vec{j}$.
(ii) Given that $10\,\vec{s} = |\overrightarrow{pr}|\overrightarrow{pq} + |\overrightarrow{pq}|\overrightarrow{pr}$, express $\vec{s}$ in terms of $\vec{i}$ and $\vec{j}$.
(iii) Find the measure of the angle between $\vec{s}$ and $\overrightarrow{pr}$.

(i) $\overrightarrow{pq} = \vec{q} - \vec{p} = \vec{i} - 6\vec{j} + 5\vec{i} - 2\vec{j} = \underline{\underline{6\vec{i} - 8\vec{j}}}.$
$\overrightarrow{pr} = \vec{r} - \vec{p} = -\vec{i} + 5\vec{j} + 5\vec{i} - 2\vec{j} = \underline{\underline{4\vec{i} + 3\vec{j}}}.$

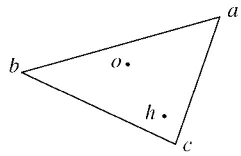
(ii) $|\overrightarrow{pr}| = |4\vec{i} + 3\vec{j}| = \sqrt{16+9} = 5$ and $|\overrightarrow{pq}| = |6\vec{i} - 8\vec{j}| = \sqrt{36+64} = 10.$
$\therefore 10\vec{s} = 5(6\vec{i} - 8\vec{j}) + 10(4\vec{i} + 3\vec{j})$
$\Rightarrow 2\vec{s} = 6\vec{i} - 8\vec{j} + 8\vec{i} + 6\vec{j} = 14\vec{i} - 2\vec{j}.$
$\therefore \vec{s} = \underline{\underline{7\vec{i} - \vec{j}}}.$

(iii) $\vec{s} \cdot \overrightarrow{pr} = (7\vec{i} - \vec{j})(4\vec{i} + 3\vec{j}) = |7\vec{i} - \vec{j}| \cdot |4\vec{i} + 3\vec{j}| \cos\theta.$

$$\therefore \cos\theta = \frac{28 - 3}{\sqrt{50}\sqrt{25}} = \frac{25}{25\sqrt{2}} = \frac{1}{\sqrt{2}}.$$

$$\therefore \underline{\underline{\theta = \frac{\pi}{4}}}.$$

(c) The origin o is the circumcentre of the triangle abc.
If $\vec{h} = \vec{a} + \vec{b} + \vec{c}$, show that $\vec{ah} \perp \vec{bc}$.



o is the circumcentre $\Rightarrow |\vec{a}| = |\vec{b}| = |\vec{c}|$.

$$\vec{ah} \perp \vec{bc} \text{ if } \vec{ah} \cdot \vec{bc} = 0$$

$$\text{if } (\vec{h} - \vec{a}) \cdot (\vec{b} - \vec{c}) = 0$$

$$\text{if } (\vec{b} + \vec{c}) \cdot (\vec{b} - \vec{c}) = 0 \ \ldots \qquad \vec{h} = \vec{a} + \vec{b} + c \quad \Rightarrow \vec{h} - \vec{a} = \vec{b} + c$$

$$\text{if } \vec{b}^2 - \vec{c}^2 = 0$$

$$\text{if } |\vec{b}|^2 - |\vec{c}|^2 = 0 \ \ldots \text{ which is true}$$

$$\Rightarrow \vec{ah} \perp \vec{bc}$$

3. (a) Show that the line containing the points (3, −6) and (−7, 12) is perpendicular to the line $5x - 9y + 6 = 0$.

Slope of line containing points (3, −6) and (−7, 12) is

$$m_1 = \frac{12 + 6}{-7 - 3} = \frac{18}{-10} = -\frac{9}{5}.$$

The line $5x - 9y + 6 = 0$ has slope $m_2 = \frac{5}{9}$.

But $m_1 \cdot m_2 = -1$, $\therefore$ lines perpendicular.

(b) The line K has positive slope and passes through the point p (2, −9).
K intersects the x-axis at q and the y-axis at r and $|pq|:|pr| = 3:1$.
Find the co-ordinates of q and the co-ordinates of r.

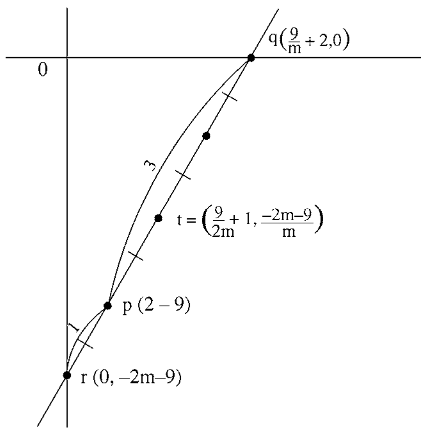
Let m the be the slope of K.

Equation of K: $y + 9 = m(x - 2)$

i.e $\quad mx - y - 2m - 9 = 0$

$$q = \left(\frac{2m + 9}{m},\ 0\right) \quad \text{i.e.,} \left(\frac{9}{m} + 2,\ 0\right)$$

$$r = (0,\ -2m - 9)$$



$|qp| : |pr| = 3 : 1$

$\Rightarrow$ t is the midpoint of $|rq|$

$$\Rightarrow t = \left(\frac{9}{2m} + 1,\ \frac{-2m - 9}{2}\right)$$

But t is also the image of r under the translation $\vec{rp}$.

$\vec{rp} = (0,\ -2m-9) \to (2,-9)$

$\Rightarrow p(2,\ -9) \to (4,\ 2m-9) = t$

$\Rightarrow (4,\ 2m-9) = \left(\dfrac{9}{2m}+1, \dfrac{-2m-9}{2}\right)$

$\Rightarrow\ 4 = \dfrac{9}{2m}+1$ or $2m-9 = \dfrac{-2m-9}{2}$

$\Rightarrow 8m = 9+2m$ or $4m-18 = -2m-9$

$\Rightarrow 6m = 9$ $\Rightarrow 6m = 9$

$\Rightarrow m = \dfrac{3}{2}$ $\Rightarrow m = \dfrac{9}{6} = \dfrac{3}{2}$

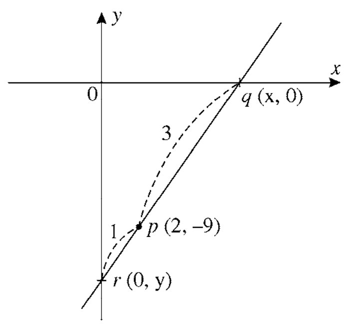
i.e. $m = \dfrac{3}{2}$

$q = \left(\dfrac{9}{m}+2,\ 0\right) \Rightarrow q = \left(9 \div \dfrac{3}{2}+2,\ 0\right) = (8,\ 0)$

$r = (0,\ -2m\ -9) \Rightarrow r\ = \left(0,\ -2\left(\dfrac{3}{2}\right)-9\right) = (0,\ -12)$

$\therefore\ q = (8,0)$ and $r\ = (0,-12)$

OR



p divides $[rq]$ in the ratio 1:3
$r = (0,\ y)$; $q = (x,\ 0)$ $h{:}k = 1{:}3$
$(x_1,\ y_1) = (x,\ 0)$; $(x_2,\ y_2) = (0,\ y)$
Let $p = (x,\ y) = (2,\ -9)$

Formula for dividing a line in a given ratio $h{:}k$

$x = \dfrac{hx_2 + kx_1}{h+k}$ $y = \dfrac{hy_2 + ky_1}{h+k}$

$\Rightarrow 2 = \dfrac{1(x) + 3(0)}{4}$ $\Rightarrow -9 = \dfrac{1(0) + 3y}{4}$

$\Rightarrow x = 8$ $\Rightarrow -36 = 3y$

$\Rightarrow y = -12$

Therefore, $q = (8,\ 0)$ and $r = (0,\ -12)$

(c) (i) Prove that the measure of one of the angles between two lines with slopes m_1 and m_2 is given by

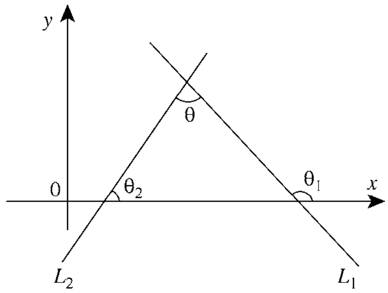
$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}.$$

(ii) L is the line $y = 4x$ and K is the line $x = 4y$.
f is the transformation $(x, y) \to (x', y')$, where $x' = 2x - y$ and $y' = x + 3y$.
Find the measure of the acute angle between $f(L)$ and $f(K)$, correct to the nearest degree.

(i) Slope $L_1 = m_1 = \tan\theta_1$.
Slope $L_2 = m_2 = \tan\theta_2$.
$\theta_1 = \theta + \theta_2 \Rightarrow \theta = \theta_1 - \theta_2$.

$$\therefore \tan\theta = \tan(\theta_1 - \theta_2) = \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2}$$

$$\therefore \tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}.$$



(ii)

$$3x' = 6x - 3y$$
$$y' = x + 3y$$

$$7x = 3x' + y' \Rightarrow x = \frac{1}{7}(3x' + y')$$

But $y' = x + 3y \Rightarrow y' = \frac{1}{7}(3x' + y') + 3y.$

$$\therefore y = \frac{1}{7}(-x' + 2y')$$

$f(L)$: $\frac{1}{7}(-x' + 2y') = \frac{4}{7}(3x' + y')$. $\therefore$ $f(L)$: $2y' = -13x' \Rightarrow$ slope $f(L) = -\frac{13}{2}$.

$f(K)$: $\frac{1}{7}(3x' + y') = \frac{4}{7}(-x' + 2y')$. $\therefore$ $f(K)$: $y' = x' \Rightarrow$ slope $f(K) = 1$.

$$\tan\theta = \frac{-\frac{13}{2} - 1}{1 - \frac{13}{2}} = \frac{15}{11} \Rightarrow \theta = 54°.$$

4. (a) Write down the values of A for which

$$\cos A = \frac{1}{2}, \text{ where } 0° \le A \le 360°.$$

$\cos A = \frac{1}{2} \Rightarrow A = 60°$ or $300°$

(b) (i) Express $\sin(3x+60°)-\sin x$ as a product of sine and cosine.

(ii) Find all the solutions of the equation

$$\sin(3x+60°)-\sin x=0, \text{ where } 0° \le x \le 360°.$$

(i) $\sin(3x+60°)-\sin x = \underline{\underline{2\cos(2x+30°)\sin(x+30°)}}$

(ii) $\sin(3x+60°)-\sin x=0$

$\Rightarrow 2\cos(2x+30°)\sin(x+30°)=0$

$\Rightarrow \cos(2x+30°)=0$ or $\sin(x+30°)=0$

$\Rightarrow 2x+30° = 90°, 270°, 450°, 630°, \ldots$

$\Rightarrow x = 30°,\ 120°,\ 210°,\ 300°$

$\sin(x+30°)=0$

$\Rightarrow x+30° = 180°,\ 360°$

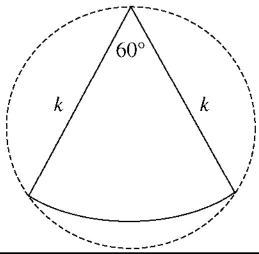
$\Rightarrow x = 150°,\ 330°$

Solution is $\underline{\underline{x = 30°,\ 120°,\ 150°,\ 210°,\ 300°,\ 330°}}$

(c) The diagram shows a sector (solid line) circumscribed by a circle (dashed line)

(i) Find the radius of the circle in terms of k.

(ii) Show that the circle encloses an area which is double that of the sector.



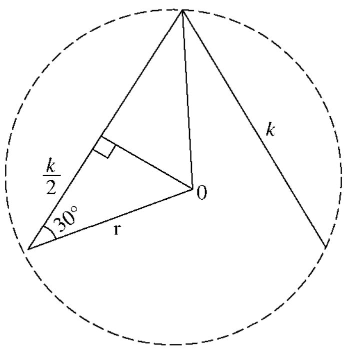
(i) $\cos 30° = \dfrac{\frac{k}{2}}{r}$ i.e. $\dfrac{k}{2r}$

But $\cos 30° = \dfrac{\sqrt{3}}{2}$

$\Rightarrow \dfrac{k}{2r} = \dfrac{\sqrt{3}}{2}$

$\Rightarrow \dfrac{k}{r} = \dfrac{\sqrt{3}}{1}$

$\Rightarrow r\sqrt{3} = k \Rightarrow \underline{\underline{r = \dfrac{k}{\sqrt{3}}}}$



(ii) Area of circle $= \pi r^2$

$$= \pi\left(\frac{k}{\sqrt{3}}\right)^2 = \frac{\pi k^2}{3}$$

Area of sector $= \dfrac{1}{2}k^2\theta$

$$= \frac{1}{2}k^2\left(\frac{\pi}{3}\right) \ldots \left(\theta = \frac{\pi}{3}\right)$$

$$= \frac{\pi k^2}{6}$$

Since $\dfrac{\pi k^2}{3} = 2\left(\dfrac{\pi k^2}{6}\right)$, area of circle is double area of the sector.

5. (a) (i) Copy and complete the table below for $f: x \to \tan^{-1}x$, giving the values for $f(x)$ in terms of π

x	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$
$f(x)$						$\frac{\pi}{4}$	

(ii) Draw the graph of $y = f(x)$ in the domain $-2 \le x \le 2$, scaling the y-axis in terms of π.

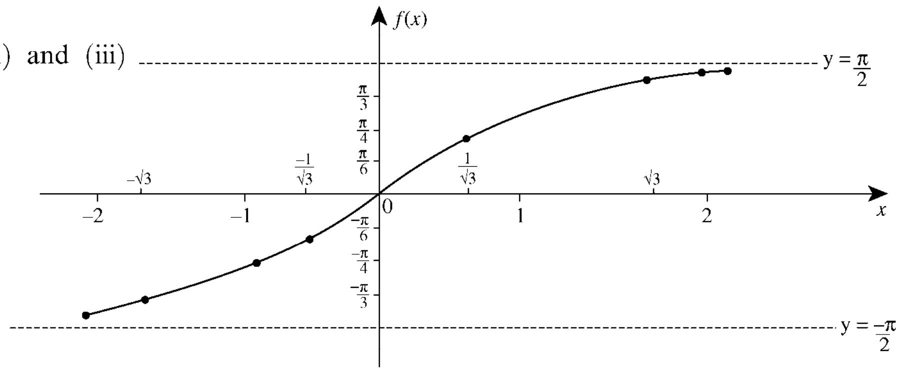
(iii) Draw the two horizontal asymptotes of the graph.

(iv) For some values of $k \in R$, but not all values, $\tan^{-1}(\tan k) = k$.
State the range of values of k for which $\tan^{-1}(\tan k) = k$.
Show, by means of an example, what happens outside the range.

(i)

x	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$
$f(x)$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$

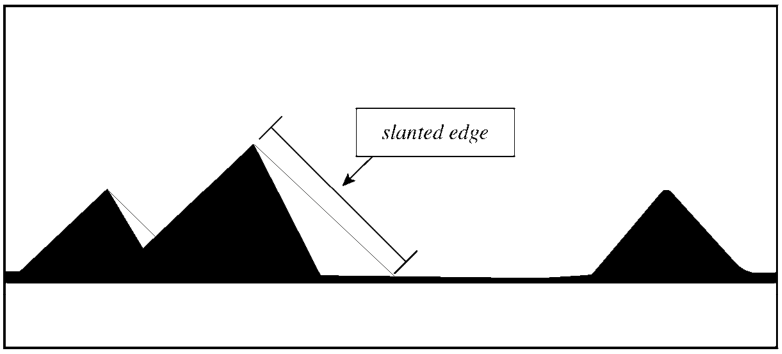
(ii) and (iii)



(iv) Range of values are $-\frac{\pi}{2} < k < \frac{\pi}{2}$.

$\tan^{-1}(\tan\frac{5\pi}{4}) = \tan^{-1}1 = \frac{\pi}{4}$

(b) The great pyramid at Giza in Egypt has a square base and four triangular faces. The base of the pyramid is of side 230 meters and the pyramid is 146 metres high. The top of the pyramid is directly above the centre of the base.

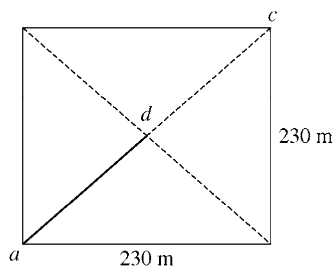


(i) Calculate the length of one of the slanted edges, correct to the nearest metre.

(ii) Calculate , correct to two significant figures, the total area of the four triangular faces of the pyramid (assuming they are smooth flat surfaces).

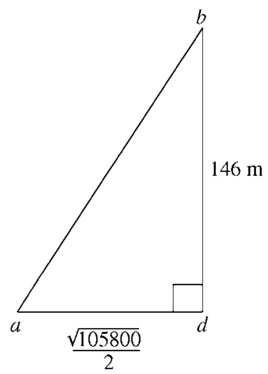
(i) The diagonal of the square base is [*ac*].

$$|ac|^2 = 230^2 + 230^2$$
$$= 105,800$$
$$\Rightarrow |ac| = \sqrt{105,800} \text{ metres}$$
$$\Rightarrow |ad| = \frac{\sqrt{105800}}{2}$$



The side $|ad|$ will form the base of a right-angled triangle of height 146 m.
The diagonal [*ab*] is the slant height of the pyramid.

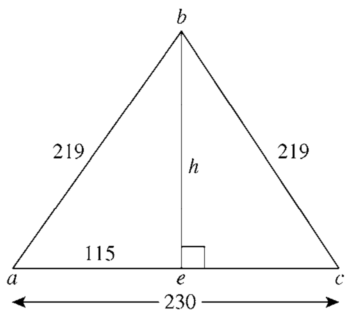
$$|ab|^2 = |ad|^2 + |db|^2$$
$$= \frac{105800}{4} + (146)^2$$
$$= 47766$$
$$\Rightarrow |ab| = \sqrt{47766} = 218.55 \text{ metres}$$
$$= \underline{\underline{219 \text{ metres}}}$$



(ii) The triangle *abc* represents one face of the pyramid.

$be \perp ac$ and $|ae| = 115$ m

$$h^2 = 219^2 - 115^2$$
$$= 34736$$
$$\Rightarrow h = 186.37 \text{ m}$$



$$\text{Area of face } abc = \frac{1}{2}(230)(186.37)$$
$$= 21432.5$$
$$\text{Area of 4 faces} = 21432.5 \times 4$$
$$= 85730$$
$$= \underline{\underline{86{,}000 \text{ m}^2}}$$

6.

> (a) (i) How many different teams of three people can be chosen from a panel of six boys and five girls?
> (ii) If the team is chosen at random, find the probability that it consists of girls only?

(i) Number of teams $= \binom{11}{3} = \frac{11 \times 10 \times 9}{3 \times 2 \times 1} = \underline{\underline{165}}$

(ii) 3 girls can be chosen from 5 girls in $\binom{5}{3}$ ways

$$\binom{5}{3} = \frac{5 \times 4 \times 3}{3 \times 2 \times 1} = 10$$

$$\text{Probability (all girls)} = \frac{10}{165} = \underline{\underline{\frac{2}{33}}}$$

> (b) (i) Solve the difference equation $6u_{n+2} - 7u_{n+1} + u_n = 0$, where $n \geq 0$, given that $u_0 = 8$ and $u_1 = 3$.
>
> (ii) Verify that the solution to part **(i)** also satisfies the difference equation $6u_{n+1} - u_n - 10 = 0$.

(i) $6u_{n+2} - 7u_{n+1} + u_n = 0$

$\Rightarrow 6x^2 - 7x + 1 = 0 \ldots$ (characteristic equation)

$\Rightarrow (x-1)(6x-1) = 0$

$\Rightarrow x = 1$ or $x = \frac{1}{6}$

$$u_n = l(1)^n + m\left(\frac{1}{6}\right)^n = l + m\left(\frac{1}{6}\right)^n$$

$u_0 = 8 \Rightarrow l + m = 8 \quad \ldots \quad ①$

$u_1 = 3 \Rightarrow l + \frac{1}{6}m = 3 \quad \ldots \quad ②$

From ① $l = 8 - m$

② $8 - m + \frac{m}{6} = 3$

$-m + \frac{m}{6} = -5$

$-6m + m = -30$

$-5m = -30 \Rightarrow m = 6$

From ①: $l = 8 - 6 \Rightarrow l = 2$

$\therefore u_n = 2(1)^n + 6\left(\frac{1}{6}\right)^n$

$$\Rightarrow \underline{\underline{u_n = 2 + \left(\frac{1}{6}\right)^{n-1}}}$$

(ii) $u_n = 2 + \left(\frac{1}{6}\right)^{n-1}$ and $6u_{n+1} - u_n - 10 = 0$

$u_{n+1} = 2 + \left(\frac{1}{6}\right)^{n}$

Substituting for u_{n+1} and $u_n$we get.

$$6\left[2 + \left(\frac{1}{6}\right)^{n}\right] - \left[2 + \left(\frac{1}{6}\right)^{n-1}\right] - 10 = 0\ldots ?$$

$$= 12 + \left(\frac{1}{6}\right)^{n-1} - 2 - \left(\frac{1}{6}\right)^{n-1} - 10$$

$$= 12 - 12 + \left(\frac{1}{6}\right)^{n-1} - \left(\frac{1}{6}\right)^{n-1}$$

$$= 0$$

$\therefore$ u_n satisfies the equation $6u_{n+1} - u_n - 10 = 0$

> (c) There are thirty days in June. Seven students have their birthdays in June. The birthdays are independent of each other and all dates are equally likely.
> (i) What is the probability that all seven students have the same birthday?
>
> (ii) What is the probability that all seven students have different birthdays?
>
> (iii) Show that the probability that at least two have the same birthday is greater than 0.5.

(i) P (all 7 have the same birthday):

$$\text{Probability} = 1 \times \frac{1}{30} \times \frac{1}{30} \times \frac{1}{30} \times \frac{1}{30} \times \frac{1}{30} \times \frac{1}{30}$$
$$= \frac{1}{(30)^6}$$

(ii) P (all 7 have different birthdays):

$$\text{Probability} = 1 \times \frac{29}{30} \times \frac{28}{30} \times \frac{27}{30} \times \frac{26}{30} \times \frac{25}{30} \times \frac{24}{30}$$

$$= \frac{2639}{5625}$$

(iii) Probability $= 1 -$ P (all 7 have different birthdays)

$$= 1 - \frac{2639}{5625} \cdots \text{from (ii) above}$$

$$= 1 - 0.4692$$

$$= 0.5308 > 0.5$$

7.

(a) The password for a mobile phone consists of five digits.

(i) How many passwords are possible?

(ii) How many of these passwords start with a 2 and finish with an odd digit?

(i) Using boxes: □ □ □ □ □

$$\text{Number of passwords} = 10 \times 10 \times 10 \times 10 \times 10$$
$$= \underline{\underline{100,000}}$$

(ii) [1] [10] [10] [10] [5]

The first 'box' can be filled in one way. (i.e. 2)
The last 'box' can be filled in 5 ways.
Each of the other 'boxes' can be filled in 10 ways.

$\therefore$ Number of passwords $= 1 \times 10 \times 10 \times 10 \times 5 = \underline{\underline{5000}}$

(b) For a lottery, 35 cards numbered 1 to 35 are placed in a drum.
Five cards will be chosen at random from the drum as the winning combination.

(i) How many different combinations are possible?

(ii) How many of all the possible combinations will match exactly four numbers with the winning combination?

(iii) How many of all the possible combinations will match exactly three numbers with the winning combination?

(iv) Show that the probability of matching at least three numbers with the winning combination is approximately 0.014.

(i) Number of Combinations $= \binom{35}{5}$

$$= \frac{35 \times 34 \times 33 \times 32 \times 31}{5 \times 4 \times 3 \times 2 \times 1}$$

$$= \underline{\underline{324,632}}.$$

(ii) Number of combinations $= \binom{5}{4} \times \binom{30}{1}$

$$= \binom{5}{1} \times \binom{30}{1}$$

$$= 5 \times 30 = \underline{\underline{150}}$$

(iii) Match three $= \binom{5}{3} \times \binom{30}{2}$ $\quad = \frac{5 \times 4 \times 3}{3 \times 2 \times 1} \times \frac{30 \times 29}{2 \times 1}$

$$= \underline{\underline{4350}}$$

(iv) P (matching at least three numbers)

$= P\text{(matching 3)} + P\text{(matching 4)} + P\text{(matching 5)}$

$$= \frac{4350 + 150 + 1}{324632} = \frac{4501}{324632}$$

$$= 0.138$$

$$= \underline{\underline{0.014}}$$

> (c) The mean of the integers from $-n$ to n, inclusive, is 0.
>
> Show that the standard deviation is $\sqrt{\dfrac{n(n+1)}{3}}$.

$$\sigma^2 = \frac{(-n)^2 + (-n+1)^2 + \cdots(-2)^2 + (-1)^2 + (0)^2 + (1)^2 \cdots + (n-2)^2 + (n-1)^2 + (n)^2}{2n+1}$$

(Note: From $-n$ to n there are $(2n+1)$ numbers.)

$$\therefore \sigma^2 = \frac{n^2 + (n^2 - 2n + 1) + \cdots + 4 + 1 + 0 + 1 + 4 + \cdots + (n^2 - 2n + 1) + n^2}{2n+1}$$

$$= \frac{2[1^2 + 2^2 + 3^2 + \cdots + n^2]}{2n+1}$$

$$= \frac{2\sum n^2}{2n+1} = \frac{2 \cdot \frac{n}{6}(n+1)(2n+1)}{2n+1}$$

$$= \frac{2n(n+1)(2n+1)}{6(2n+1)}$$

$$= \frac{2n(n+1)}{6} = \frac{n(n+1)}{3}$$

$$\Rightarrow \sigma = \underline{\underline{\sqrt{\frac{n(n+1)}{3}}}}$$

8. (a) Derive the Maclaurin series for $f(x) = e^x$ up to and including the term containing x^3.

$$f(x) = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \cdots$$

$f(x) = e^x \Rightarrow f(0) = e^0 = 1.$

$f'(x) = e^x \Rightarrow f'(0) = 1.$

$f''(x) = e^x \Rightarrow f''(0) = 1.$

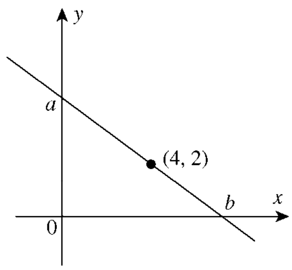
$f'''(x) = e^x \Rightarrow f'''(0) = 1.$

$$\therefore\ f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

(b) A line passes through the point (4,2) and has slope m, where $m < 0$. The line intersects the axes at the points a and b.

(i) Find the co-ordinates of a and b, in terms of m.

(ii) Hence, find the value of m for which the area of triangle aob is a minimum.



(i) Equation of ab : $y - 2 = m(x - 4)$

$$y - 2 = mx - 4m$$

$$\Rightarrow mx - y - 4m + 2 = 0$$

$$\therefore\ a = (0, -4m + 2) \text{ and } b = \left(\frac{4m-2}{m}, 0\right)$$

$$\Rightarrow b = \left(4 - \frac{2}{m}, 0\right)$$

(ii) Area of $\Delta oab = \frac{1}{2}|ob|.|oa|$

$$= \frac{1}{2}\left(4 - \frac{2}{m}\right)(-4m + 2)$$

$$= \frac{1}{2}\left(-16m + 8 + 8 - \frac{4}{m}\right)$$

$$= \frac{1}{2}\left(-16m + 16 - \frac{4}{m}\right)$$

$$\Rightarrow \text{Area, A} = -8m + 8 - \frac{2}{m}$$

$$\text{A} = -8m + 8 - 2m^{-1}$$

$$\frac{dA}{dm} = -8 + 2m^{-2} = 0, \text{ for minimum}$$

$$\Rightarrow -8m + \frac{2}{m^2} = 0$$

$$\Rightarrow -8m^2 + 2 = 0$$

$$\Rightarrow 4m^2 = 1 \Rightarrow m^2 = \frac{1}{4}$$

$$\Rightarrow m = -\frac{1}{2} \text{ as } m < 0$$

$$\text{Area} = -8m + 8 - \frac{2}{m}$$

$$= -8(-\frac{1}{2}) + 8 - \frac{2}{-\frac{1}{2}}$$

$$= 4 + 8 + 4$$

Area = 16, … minimum area

> (c) Use the ratio test to test each of the following series for convergence.
> In each case, specify clearly the range of values of x for which the series converges, the range of values for which it diverges, and the value(s) of x for which the test is inconclusive.
>
> (i) $\sum_{n=1}^{\infty} n3^n x^n$ (ii) $\sum_{n=1}^{\infty} \frac{(n+1)!n!}{(2n)!} x^n$.

(i) $u_n = n3^n x^n \Rightarrow u_{n+1} = (n+1)3^{n+1}x^{n+1}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)3^{n+1}x^{n+1}}{n3^n x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)3x}{n}\right| = \lim_{x\to\infty}\left|\left(1+\frac{1}{n}\right)3x\right|$$

$$= |3x|$$

The series converges for $|3x| < 1$

i.e $|x| < \frac{1}{3}$

$\Rightarrow -\frac{1}{3} < x < \frac{1}{3}$

The series diverges for $|3x| > 1$

i.e. $|x| > \frac{1}{3}$

$\Rightarrow x > \frac{1}{3}$ or $x < -\frac{1}{3}$

The series is inconclusive for $|3x| = 1$

i.e. $x = \pm\frac{1}{3}$

(ii) $\sum_{n=1}^{\infty} \frac{(n+1)!n!}{(2n)!} x^n$

$$u_n = \frac{(n+1)!n!}{(2n)!}x^n \Rightarrow u_{n+1} = \frac{(n+2)!(n+1)!}{(2n+2)!}x^{n+1}$$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(n+2)!(n+1)!}{(2n+2)!}x^{n+1} \times \frac{(2n)!}{(n+1)!n!x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+2)(n+1)x}{(2n+2)(2n+1)}\right|$$

$$= \lim_{n\to\infty}\left|\frac{n^2+3n+1}{4n^2+6n+2}x\right|$$

$$= \lim_{x\to\infty}\left|\frac{1+\frac{3}{n}+\frac{1}{n^2}}{4+\frac{6}{n}+\frac{2}{n^2}}x\right|$$

$$=\left|\frac{1}{4}\cdot x\right|=\left|\frac{x}{4}\right|$$

The series converges for $\left|\frac{x}{4}\right| < 1$

i.e. $|x| < 4$

$\Rightarrow \underline{\underline{-4 < x < 4}}$

The series diverges for $\left|\frac{x}{4}\right| > 1$

i.e. $|x| > 4$

$\Rightarrow \underline{\underline{x > 4 \text{ or } x < -4}}$

The series is inconclusive for $\left|\frac{x}{4}\right| = 1$

i.e. $|x| = 4$

$\Rightarrow \underline{\underline{x = \pm 4}}$

9. (a) z is a random variable with standard normal distribution.
Find the value of z_1 for which $P(z > z_1) = 0.0808$.

$P(z > z_1) = 0.0808 \Rightarrow 1 - P(z < z_1) = 0.0808$

$\therefore\ P(z < z_1) = 0.9192$

$\Rightarrow \underline{\underline{z_1 = 1.4}}$

(b) A bag contains the following cardboard shapes:
10 red squares, 15 green squares, 8 red triangles and 12 green triangles.
One of the shapes is drawn at random from the bag.
E is the event that a square is drawn.
F is the event that a green shape is drawn.

(i) Find $P(E \cap F)$.

(ii) Find $P(E \cup F)$.

(iii) State whether E and F are independent events, giving a reason for your answer.

(iv) State whether E and F are mutually exclusive events, giving a reason for your answer.

(i) $E \cap F =$ green squares

$$P(E\cap F) = \frac{15}{45} = \underline{\underline{\frac{1}{3}}}$$

(ii) $E \cup F =$ Squares or green shapes

$= (10 + 15 + 12)$

$= 37$

$$P(E\cup F) = \underline{\underline{\frac{37}{45}}}$$

OR

$$P(E\cup F) = P(E) + P(F) - P(E\cap F)$$

$$= \frac{25}{45} + \frac{27}{45} - \frac{15}{45}$$

$$= \underline{\underline{\frac{37}{45}}}$$

(iii) $P(E).P(F)=\frac{25}{45}\cdot\frac{27}{45}=\frac{15}{45}=\frac{1}{3}$

$P(E\cap F)=\frac{1}{3}\cdots$ from (i) above

Since $P(E).P(F)=P(E\cap F)=\frac{1}{3}$, the events are independent.

(iv) $P(E\cap F)\neq 0 \Rightarrow$ E and F are not mutually exclusive.

(c) The marks awarded in an examination are normally distributed with a mean mark of 60 and a standard deviation of 10.
A sample of 50 students has a mean mark of 63.
Test, at the 5% level of significance, the hypothesis that this is random sample from the population.

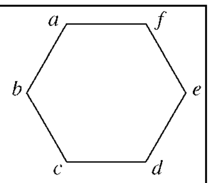
$\overline{x}=60,\ \ \sigma=10\Rightarrow\sigma_{\overline{x}}=\frac{\sigma}{\sqrt{n}}=\frac{10}{\sqrt{50}}=1.4142$

$\frac{x-\overline{x}}{\sigma_{\overline{x}}}=\frac{63-60}{1.4142}=2.1213$

Since 2.1213 > 1.96, the sample is not random.

10. (a) G is the set of rotations that map a regular hexagon onto itself. $(G, \circ)$ is a group , where $\circ$ denotes composition.

The anti-clockwise rotation through 60° is written as R_{60°.



(i) List the elements of G.

(ii) State which elements of the group, if any, are generators.

(iii) List all the proper subgroups of $(G,\circ)$.

(iv) Find $Z(G)$, the centre of $(G,\circ)$. Justify your answer.

(i) $G=\{R_{0}, R_{60^\circ}, R_{120^\circ}, R_{180^\circ}, R_{240^\circ}, R_{300^\circ}\}$

(ii) Generators are R_{60° and R_{300°.

(iii) Proper subgroups of $(G, \circ)$ are

$\{R_{0^\circ}, R_{180^\circ}\}$ and $\{R_{0^\circ}, R_{120^\circ}, R_{240^\circ}\}$

(iv) $Z(G)$ is the set of all elements of G which commute with every element of G.
Each element of G commutes with all other elements of G,

$\Rightarrow \underline{\underline{Z(G) = G}}$

> (b) (i) Show that the group $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$ under matrix multiplication is isomorphic to the group $\{0,1\}$ under addition modulo 2.
>
> (ii) Prove that any infinite cyclic group is isomorphic to $(\mathbf{Z}, +)$.

(i) Let $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

Let $G = \{I, A\}$ under matrix multiplication.

$$A^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

Thus I is of order 1 and A is of order 2
Let $H = \{0,1\}$ under addition modulo 2.
$1+1 = 2 = 0 \pmod 2$
Thus 0 is of order 1 and 1 is of order 2.
As both G and H have one element of order 1 and one element of order 2, they are isomorphic.
The isomorphism $G \to H$ is

$$I \to 0$$
$$A \to 1$$

(ii) Let $G = \{g^n, n \in Z\}$, i.e. an infinite cyclic group under multiplication.
The bijection f: $G, * \to \mathrm{Z}, +$ is defined by $f(g^k) = k$
Let g^k and $g^l \in G$.

$$f(g^k * g^l) = f(g^{k+l})$$
$$= k + l$$
$$= f(g^k) + f(g^l)$$

$\Rightarrow f(g^k * g^l) = f(g^k) + f(g^l)$

$\Rightarrow G, *$ and $Z, +$ are isomorphic.

11.

(a) (i) Find the image of $a\ (-1,2)$ and $b\ (0,4)$ under the transformation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2 \\ -4 \end{pmatrix}.$$

(ii) Show that ab is parallel to $a'b'$

(i) $$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ -4 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \end{pmatrix} + \begin{pmatrix} 2 \\ -4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\therefore$ the image of $(-1,2)$ is $\underline{\underline{(0,0)}}$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 4 \end{pmatrix} + \begin{pmatrix} 2 \\ -4 \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \end{pmatrix} + \begin{pmatrix} 2 \\ -4 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

$\therefore$ the image of $(0,4)$ is $\underline{\underline{(2,4)}}$

(ii) Slope of $ab = \dfrac{4-2}{0+1} = 2$

Slope of $a'b' = \dfrac{4-0}{2-0} = \dfrac{4}{2} = 2$

Since slope of ab = slope of $a'b' \Rightarrow ab$ is parallel to $a'b'$

(b) $p\,(x,y)$ is a point such that the distance from p to the point $(2,0)$ is half the distance from p to the line $x = 8$.

(i) Find the equation of the locus of p.

(ii) Show that this locus is an ellipse centred at the origin, by expressing its equation in the form $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

(i) Let $p = (x,y)$ and $s = (2,0)$

$|ps| = \sqrt{(x-2)^2 + (y-0)^2}$

Distance from (x,y) to the line $x = 8$ is $|\,8 - x\,|$.

$|ps| = \dfrac{1}{2}|8-x|$

$\Rightarrow \sqrt{(x-2)^2 + y^2} = \dfrac{1}{2}|8-x|$

$\Rightarrow (x-2)^2 + y^2 = \dfrac{1}{4}(8-x)^2$

$$\Rightarrow x^2 - 4x + 4 + y^2 = \frac{1}{4}(64 - 16x + x^2)$$

$$\Rightarrow 4x^2 - 16x + 16 + 4y^2 = 64 - 16x + x^2$$

$$\Rightarrow 3x^2 + 4y^2 = 48$$

The equation of the locus of p is $\underline{\underline{3x^2 + 4y^2 = 48}}$

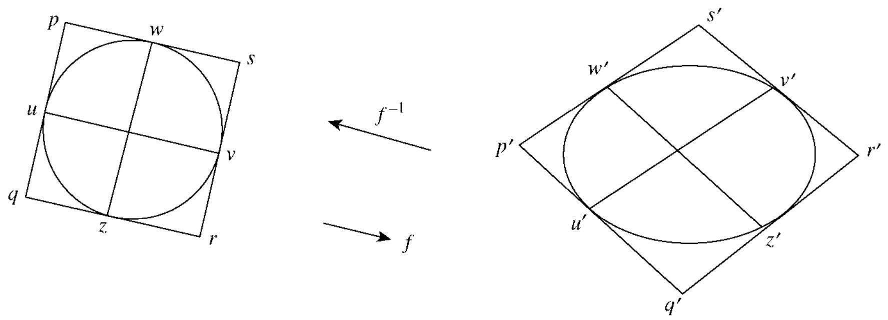
(ii) $3x^2 + 4y^2 = 48$

$$\Rightarrow \frac{3x^2}{48} + \frac{4y^2}{48} = 1$$

$$\Rightarrow \frac{x^2}{16} + \frac{y^2}{12} = 1$$

This is in the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and represents an ellipse centred at the origin.

(c)	Prove that the areas of all parallelograms circumscribed about a given ellipse at the endpoints of conjugate diameters are equal.



$[w'z']$ and $[u'v']$ are conjugate diameters of ellipse E.
Tangents at their end-points form the parallelogram $p'q'r's'$.
Under an affine transformation f^{-1},the ellipse maps to the circle $x^2 + y^2 = 1$ and $p'q'r's'$ is mapped to $pqrs$.
$[uv]$ and $[wz]$ are conjugate diameters of the circle and $uv \perp wz$.

The square $pqrs$ has fixed area 4 sq units.
Area $pqrs = 2$ area $pqr \Rightarrow$ area $p'q'r's' =$ 2 area $p'q'r'$ as ratio is an invariant map.
Area $p'q'r's' = 2|\det.f|$area Δpqr
$= |\det.f|$area $pqrs$.
But $\det.f$ is constant and area $pqrs$ is also constant $\Rightarrow$ area $p'q'r's'$ is constant.
$\therefore$ Areas of all parallelograms at end points of conjugate diameters are equal.